Section 20
Highest and lowest values of trigonometric functions

The problems of finding the largest and smallest values of any functions, including trigonometric ones, are solved mainly by means of differential calculus. But solving such problems by elementary methods has its advantages.:

1) promotes deeper development of logical thinking;

2) makes it possible to achieve success faster and easier than using a derivative.

In the process of solving this type of problem, it is useful to reduce the entire trigonometric expression to one function, and then look for its largest and smallest values.

Find the largest value of a function $y = \sin x + \cos x$ on the interval $\left(0; \frac{\pi}{2}\right)$.

Solution:

$$y = \sin x + \cos x = \sin x + \sin\left(\frac{\pi}{2} - x\right) =$$

$$= 2\sin\frac{x + \frac{\pi}{2} - x}{2} \cdot \frac{x - \frac{\pi}{2} + x}{2} =$$

$$= 2\sin\frac{\pi}{4} \cdot \cos\left(x - \frac{\pi}{4}\right) = 2\sin\frac{\pi}{4} \cdot \cos\left(x - \frac{\pi}{4}\right) =$$

$$= 2 \cdot \frac{\sqrt{2}}{2} \cdot \cos\left(x - \frac{\pi}{4}\right) = \sqrt{2} \cdot \cos\left(x - \frac{\pi}{4}\right).$$

Casting formulas.
Formulas for converting the sum of trigonometric functions into a product.

From the condition $0 < x < \frac{\pi}{2}$ evaluate expression $x - \frac{\pi}{4}$:

$$0 < x < \frac{\pi}{2} \Big| -\frac{\pi}{4}, \quad -\frac{\pi}{4} < x - \frac{\pi}{4} < \frac{\pi}{2} - \frac{\pi}{4}; \quad -\frac{\pi}{4} < x < \frac{\pi}{4}.$$

Function values $y = \sqrt{2} \cdot \cos\left(x - \frac{\pi}{4}\right)$ will be the largest when $\cos\left(x - \frac{\pi}{4}\right) = 1$, and this equality holds for $x - \frac{\pi}{4} = 0$, that is, when $x = \frac{\pi}{4}$.

Answer: $Y_{найб} = \sqrt{2}$ при $x = \frac{\pi}{4}$.

Find smallest function value $Y = \sqrt{3}\sin 2x - \cos 2x$ на $[0; \pi]$.

Solution:

Let us simplify this expression by reducing it to one function:

$$y=\sqrt{3}\sin 2x-\cos 2x=2\cdot\left(\frac{\sqrt{3}}{2}\sin 2x-\frac{1}{2}\cos 2x\right)=2\cdot\left(\cos\frac{\pi}{6}\cdot\sin 2x-\sin\frac{\pi}{6}\cdot\cos 2x\right)=$$

$$=2\cdot\sin\left(2x-\frac{\pi}{6}\right).$$

Based on the condition $0\le x\le\pi,$ evaluate expression $2x-\frac{\pi}{6}$:

$0\le x\le\pi\,\big|\cdot 2,\ 0\le 2x\le 2\pi\,\Big|-\frac{\pi}{6};\ -\frac{\pi}{6}\le 2x-\frac{\pi}{6}\le 1\frac{5}{6}\pi;\ Y=2\cdot\sin\left(2x-\frac{\pi}{6}\right)$ takes the least value when $\sin\left(2x-\frac{\pi}{6}\right)=-1,$ that is $2x-\frac{\pi}{6}=\frac{3\pi}{2};$

$$2x=\frac{3\pi}{2}+\frac{\pi}{6}=\frac{9\pi+\pi}{6}=\frac{10\pi}{6}=1\frac{4}{6}\pi=1\frac{2}{3}\pi;\ \ x=1\frac{2}{3}\pi:2=\frac{5\pi}{3}\cdot\frac{1}{2}=\frac{5\pi}{6}.$$

Answer: $y_{найм}=-2$ at $y=\frac{5\pi}{6}.$

Find the largest value of a function $Y=\sin x+2\cos x$ at $[0;\pi]$.

Solution:

$Y=\sin x+2\cos x=1\cdot\sin x+2\cos x$

$1^2+2^2=5,$

$\sqrt{1^2+2^2}=\sqrt{5}.$

Move $\sqrt{5}$ out of the parentheses:

We denote $\frac{1}{\sqrt{5}}-\cos\varphi,\frac{2}{\sqrt{5}}=\sin\varphi.$

$$=\sqrt{5}\cdot\left(\frac{1}{\sqrt{5}}\sin x+\frac{2}{\sqrt{5}}\cos x\right)=\sqrt{5}\cdot(\cos\varphi\cdot\sin x+\sin\varphi\cdot\cos x)=\sqrt{5}\cdot\sin(x+\varphi).$$

Because the $\frac{1}{\sqrt{5}}>0,\ \frac{2}{\sqrt{5}}>0,$ then $\varphi\in\left(0;\frac{\pi}{2}\right).$

$Y=\sqrt{5}\cdot\sin(x+\varphi)$ will reach the highest value at $\sin(x+\varphi)=1,$ that is, when $x+\varphi=\frac{\pi}{2},$

$$x=\frac{\pi}{2}-\varphi=\frac{\pi}{2}-\arccos\frac{1}{\sqrt{5}}=0{,}46.$$

Answer: $y_{найб}=\sqrt{5}$ at $x=0{,}46.$

Find the largest value of a function:

$Y=\sin x\cdot\cos^3 x-\sin^3 x\cdot\cos x.$

Solution:

$$Y=\sin x\cdot\cos^3 x-\sin^3 x\cdot\cos x=\sin x\cdot\cos x\cdot\left(\cos^2 x-\sin^2 x\right)=$$

$$=\frac{1}{2}\cdot 2\cdot\sin x\cdot\cos x\cdot(\cos 2x)=\frac{1}{2}\sin 2x\cdot\cos 2x=\frac{1}{2}\cdot\frac{1}{2}\cdot 2\cdot\sin 2x\cdot\cos 2x=\frac{1}{4}\cdot\sin 4x.$$

Function $Y=\frac{1}{4}\cdot\sin 4x$ takes on the greatest importance at $\sin 4x=1.$

Answer: $y_{найб}=\frac{1}{4}.$

Find the largest value of a function $Y=9\cdot 3^{x}\cdot 3^{-x^{2}}$.

Solution:

$Y=9\cdot 3^{x}\cdot 3^{-x^{2}}=3^{2}\cdot 3^{x}\cdot 3^{-x^{2}}=3^{2+x-x^{2}}$.

Because $3>1$, then the exponential function increases.

It will reach its greatest value when the indicator $-x^{2}+x+2$ will matter most. We denote $\varphi(x)=-x^{2}+x+2$. Function graph $\varphi(x)$ is a parabola with branches downward. We find the coordinates of its vertices $(m;n)$:

$$m=-\frac{b}{2a}=-\frac{1}{2\cdot 1}=\frac{1}{2};\ n=\varphi(m)=-\left(\frac{1}{2}\right)^{2}+\frac{1}{2}+2=-\frac{1}{4}+\frac{1}{2}+2=2\frac{1}{4}.$$

$$y_{найб}=3^{2\frac{1}{4}}=3^{2}\cdot 3^{\frac{1}{4}}=9\cdot\sqrt[4]{3}.$$

Answer: $9\sqrt[4]{3}$.

Find smallest function value $Y=\log_{(x^{2}-6x+36)}\frac{1}{3}$.

Solution:

Let's move on to the base of the logarithm $\frac{1}{3}$:

$$Y=\log_{(x^{2}-6x+36)}\frac{1}{3}=\frac{\log_{\frac{1}{3}}\frac{1}{3}}{\log_{\frac{1}{3}}(x^{2}-6x+36)}=\frac{1}{\log_{\frac{1}{3}}(x^{2}-6x+36)}.$$

Smallest function value $y^{\frac{1}{3}}$ is observed when the expression $\log_{\frac{1}{3}}(x^{2}-6x+36)$ reaches its highest value. We denote $\varphi(x)=\log_{\frac{1}{3}}(x^{2}-6x+36)$.

Because the $0<\frac{1}{3}<1$, then the logarithmic function $\varphi(x)$ falls off. It reaches its highest value with the smallest argument value. We find the coordinates of the vertex of the parabola $x^{2}-6x+36=f(x)$:

$$m=-\frac{b}{2a}=\frac{6}{2}=3;\ n=f(m)=9-18+36=27.$$

$$y_{найм}=\frac{1}{\log_{\frac{1}{3}}27}=\frac{1}{\log_{\frac{1}{3}}3^{3}}=\frac{1}{3\log_{\frac{1}{3}}3}=\frac{1}{3\log_{\frac{1}{3}}3^{-1}}=\frac{1}{-3\cdot 1}=-\frac{1}{3}.$$

Answer: $-\frac{1}{3}$.

Find the largest value of a function $Y=\sin 2x\cdot\sin\left(2x-\frac{\pi}{5}\right)$.

Solution:

We transform the product of trigonometric functions into the sum:

$$Y=\sin 2x\cdot\sin\left(2x-\frac{\pi}{5}\right)=\frac{1}{2}\cdot\left(\cos\left(2x-2x+\frac{\pi}{5}\right)-\cos\left(2x+2x-\frac{\pi}{5}\right)\right)=\frac{1}{2}\left(\cos\frac{\pi}{5}-\cos\left(4x-\frac{\pi}{5}\right)\right).$$

This expression reaches its greatest value when $\cos\left(4x-\frac{\pi}{5}\right)$ reaches its lowest value, that is, $\cos\left(4x-\frac{\pi}{5}\right)<-1,$

$$y_{найб}=\frac{1}{2}\cdot\left(\cos\frac{\pi}{5}-(-1)\right)=\frac{1}{2}\left(\cos\frac{\pi}{5}+1\right).$$

We transform this expression:

$$y_{найб}=\frac{1}{2}\cdot\left(\cos\frac{\pi}{5}+1\right)=\frac{1}{2}\left(\cos\left(2\cdot\frac{\pi}{10}\right)+\sin^2\frac{\pi}{10}+\cos^2\frac{\pi}{10}\right)=\frac{1}{2}\left(\cos^2\frac{\pi}{10}-\sin^2\frac{\pi}{10}+\cos^2\frac{\pi}{10}+\sin^2\frac{\pi}{10}\right)=$$

$$=\frac{1}{2}\cdot 2\cos^2\frac{\pi}{100}=\cos^2\frac{\pi}{10}\approx 0{,}905.$$

Answer: 0,905.

Find Largest Function Value $Y=\sin\left(\frac{\pi}{3}+2x\right)\cdot\cos\left(\frac{\pi}{6}-2x\right)$ на $(0;\pi)$.

Solution:

We transform the product of trigonometric functions in the sum:

$$Y=\sin\left(\frac{\pi}{3}+2x\right)\cdot\cos\left(\frac{\pi}{6}-2x\right)=\frac{1}{2}\left(\sin\left(\frac{\pi}{3}+2x+\frac{\pi}{6}-2x\right)+\sin\left(\frac{\pi}{3}+2x-\frac{\pi}{6}+2x\right)\right)=$$

$$=\frac{1}{2}\left(\sin\frac{\pi}{2}+\sin\left(\frac{\pi}{6}+4x\right)\right)=\frac{1}{2}\left(1+\sin\left(\frac{\pi}{6}+4x\right)\right).$$

Function $y=\frac{1}{2}\left(1+\sin\left(\frac{\pi}{6}+4x\right)\right)$ reaches its greatest value when $\sin\left(\frac{\pi}{6}+4x\right)=1,$ that is, $\frac{\pi}{6}+4x=\frac{\pi}{2}$. From the condition $0\le x\le\pi$ estimate the value $4x+\frac{\pi}{6}$:

$$0\le x\le\pi\,\Big|\cdot 4$$

$$0\le 4x\le 4\pi\,\Big|+\frac{\pi}{6}$$

$$\frac{\pi}{6}\le 4x+\frac{\pi}{6}\le 4\frac{1}{6}\pi.$$

On the $\left[\frac{\pi}{6};\frac{25}{6}\pi\right]$ at $\frac{\pi}{2}$ and $\frac{\pi}{2}+2\pi=\frac{5\pi}{2}$.

Sinus reaches its maximum value,

if a $4x+\frac{\pi}{6}=\frac{\pi}{2};\ 4x=\frac{\pi}{3};\ x=\frac{\pi}{12};$

if a $4x+\frac{\pi}{6}=\frac{5\pi}{2};\ 4x=\frac{7\pi}{3};\ x=\frac{7\pi}{12}.$

In this way $y_{найб}=\frac{1}{2}(1+1)=1$ at $x=\frac{\pi}{12}$ and $x=\frac{7\pi}{12}$.

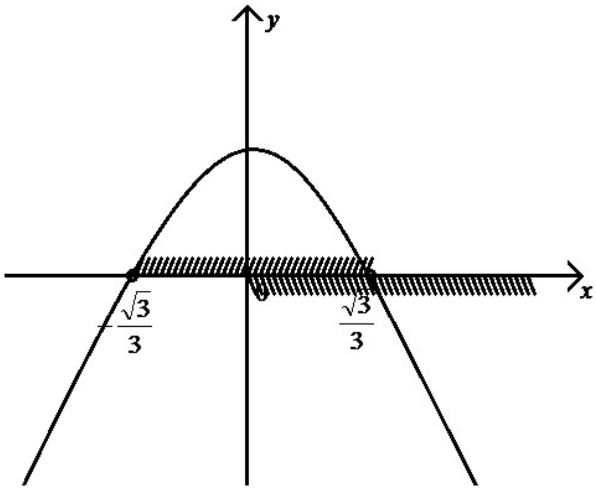
Answer: 1.

Find the largest value of a function $Y=x\cdot\sqrt{1-3x^2}$.

Solution:

We find the domain of the function $Y = x\cdot\sqrt{1-3x^2}$:

$$\begin{cases} x\ge 0, \\ 1-3x^2\ge 0 \end{cases} \begin{cases} x\ge 0, \\ 3\left(\frac{1}{3}-x^2\right)\ge 0 \end{cases} \begin{cases} x\ge 0, \\ \left(\sqrt{\frac{1}{3}}\right)^2-x^2\ge 0 \end{cases} \begin{cases} x\ge 0, \\ \left(\frac{\sqrt{3}}{3}-x\right)\cdot\left(\frac{\sqrt{3}}{3}+x\right)\ge 0. \end{cases}$$



$D(y)=\left(0;\frac{\sqrt{3}}{3}\right]$. In the field of definition $y\ge 0$.

$Y = x\cdot\sqrt{1-3x^2}=\sqrt{x^2\cdot(1-3x^2)}=\sqrt{x^2-3x^4}$.

Substitution: $x^2=t,\ \ y=\sqrt{t-3t^2}$.

This function reaches its greatest value in the domain of definition when the parabola $t^2-3t^2=\varphi(f)$ reaches its highest value.

This happens at the apex of the parabola.

$m=-\frac{1}{-3\cdot 2}=\frac{1}{6};\ \ \varphi\left(\frac{1}{6}\right)=\frac{1}{6}-3\cdot\left(\frac{1}{6}\right)^2=\frac{1}{6}-3\cdot\frac{1}{36}=\frac{1}{6}-\frac{1}{12}=\frac{1}{12};$

Returning to the replacement, we get:

$x^2=\frac{1}{6}$. Considering $D(y)$, take $x=\sqrt{\frac{1}{6}}=\frac{\sqrt{6}}{6};\ \ y=\sqrt{\frac{1}{12}}=\sqrt{\frac{1}{4\cdot 3}}=\frac{1}{2}\cdot\sqrt{\frac{3}{9}}=\frac{1}{6}\sqrt{3}.$

Answer: $\frac{1}{6}\sqrt{3}$.

A positive number *a* is represented as a sum of two terms so that their product is the largest.

Solution:

Let one term be equal to x, then the second term $a-x$.

Let us denote their product $\varphi(x)=x\cdot(a-x)=ax-x^2=-x^2+ax$.

Because $-1<0$, then the function $\varphi(x)$ reaches its maximum value at the apex of the parabola $x_0=-\frac{a}{-2}=\frac{a}{2},\ \ x_0=\frac{a}{2}.\ \ \frac{a}{2}-$ one term, $a-\frac{a}{2}=\frac{a}{2}-$ second term.

Answer: $\frac{a}{2}$ i $\frac{a}{2}$.

Self-study assignments:

Find Largest and Smallest Function Values:

$y = 2\cos x.$ Answer: $2; -2.$

$y = -5\sin x.$ Answer: $5; -5.$

$y = 3 + 2\sin x.$ Answer: $5; 1.$

$y = 2\sin x - 3.$ Answer: $-1; -5.$

$y = 5\cos x + 1.$ Answer: $6; -4.$

$y = 2 + 3\sin^2 x.$ Answer: $5; 2.$

$y = 3 - \cos^2 x.$ Answer: $3; 2.$

$y = \frac{1}{2 + \cos x}.$ Answer: $1; \frac{1}{3}.$

$y = \frac{1}{3 - \sin x}.$ Answer: $\frac{1}{2}; \frac{1}{4}.$

$y = 5\sin^2 x - 2\cos^2 x.$ Answer: $5; -2.$

$y = 3\sin x + 4\cos x.$ Answer: $5; -5.$

$y = \sin 4x - \sqrt{3}\cos 4x.$ Answer: $y_{найб} = 2,\ y_{найм} = -2.$

Find smallest function value:

а) $y = \sin x - \cos^2 x - 1;$ б) $y = \sin x - \sin\left(x + \frac{\pi}{4}\right).$

Answer: а) $y_{найм} = -2\frac{1}{4};$ б) $y_{найм} = -2\sin\frac{\pi}{8}.$

Find Largest and Smallest Function Values:

а) $y = \frac{1}{x^2 + x + 1}$ at $[-1; 1]$

Answer: $y_{найб} = \frac{4}{3}, y_{найм} = \frac{1}{3}.$

б) $y = x^4 + 3x^2 + 2$ at $[-2; 3]$

Answer: $y_{найб} = 110, y_{найм} = 2.$

в) $y = \sin^2 x + \sin x + 1.$

Answer: $y_{найб} = 3, y_{найм} = \frac{3}{4}.$

Finding the period of trigonometric functions

The period of a function is the smallest argument value after which it repeats all of its values. Denoted by the letter T.

Functions $y = \sin x$ and $y = \cos x$ have a period $T = 2\pi.$

Functions $y = tgx$ and $y = ctgx$ have a period $T = \pi.$

If a $T-$ period of the function, then the number $nT,$ where $n \in N-$ also function period.

Advice:

1) To find the period of a function $f(kx),$ function period follows $f(x)$ split into $K.$
2) To find the period of a function that is the sum of functions $f(x)$ and $g(x),$ We need to find their periods T_1 and $T_2,$ and then determine the least common multiple of these periods, i.e. $T = HCK(T_1;T_2).$

Find the period of a function $\varphi(x) = \sin 2x + \cos 4x.$

Solution:

$\sin 2x = \sin kx, k = 2.$ Sinus period $2\pi.$

Therefore, the period $\sin 2x$ $T_1 = \frac{2\pi}{2} = \pi.$

Similarly, we find the period $\cos 4x:$

$T_2 = \frac{2\pi}{4} = \frac{\pi}{2}.$ Find the period of the function $\varphi(x):$

$T = HCK\left(\pi; \frac{\pi}{2}\right) = \pi.$

Answer: $\pi.$

$f(x) = 3\sin 4x + 6\sin x + \sin(x - \pi) + 5\sin(x + \pi).$

Solution:

Using the odd sine and the reduction formula, we simplify this function:

$f(x) = 3\sin 4x + 6\sin x - \sin(\pi - x) + 5\sin(\pi + x) = 3\sin 4x + 6\sin x - \sin x - 5\sin x = 3\sin 4x.$

The number 3 does not affect the period $K = 4.$ $T = \frac{2\pi}{4} = \frac{\pi}{2}.$

Answer: $\frac{\pi}{2}.$

$Y = \sin 2x + tg\frac{x}{2}.$

Solution:

$T_1 = \frac{2\pi}{2} = \pi.$ $T_2 = \frac{\pi}{\frac{1}{2}} = 2\pi.$ $T = HCK(\pi; 2\pi) = 2\pi.$

Answer: $2\pi.$

$\varphi(x) = tg\frac{x}{5} + \cos\frac{x}{3}.$

Solution:

$T_1 = \frac{\pi}{\frac{1}{5}} = 5\pi,$ $T_2 = \frac{2\pi}{\frac{1}{3}} = 6\pi.$ $T = HCK(5\pi; 6\pi) = 30\pi.$

Answer: $30\pi.$

$Y = \sin\frac{3x}{4} + 5\cos\frac{2x}{3}.$

Solution:

$$T_1=\frac{2\pi}{\frac{3}{4}}=\frac{8\pi}{3},\;\; T_2=\frac{2\pi}{\frac{2}{3}}=3\pi.\;\; T=HCK\left(\frac{8\pi}{3};3\pi\right)=24\pi.$$

Answer: $24\pi.$

$Y=\cos^2 x.$

Solution:

$$y=\cos^2 x=\frac{1+\cos 2x}{2}.\; T=\frac{2\pi}{2}\pi.$$

Answer: $\pi.$

$Y=\sin\frac{\pi x}{3}+\sin\frac{\pi x}{2}.$

Solution:

$$T_1=\frac{2\pi}{\frac{\pi}{3}}=\frac{2\pi}{1}\cdot\frac{3}{\pi}=6;\quad T_2=\frac{2\pi}{\frac{\pi}{2}}=4.\;\; T=HCK(6;4)=12.$$

Answer: $12.$

$Y=\dfrac{1}{tg\left(\frac{x}{\sqrt{2}}-\frac{\pi}{4}\right)}.$

Solution:

$$Y=\frac{1}{tg\left(\frac{x}{\sqrt{2}}-\frac{\pi}{4}\right)}=\frac{1}{\dfrac{tg\frac{x}{\sqrt{2}}-tg\frac{\pi}{4}}{1+\frac{x}{\sqrt{2}}\cdot tg\frac{\pi}{4}}}=\frac{1+tg\frac{x}{\sqrt{2}}\cdot 1}{tg\frac{x}{\sqrt{2}}-1};\;\; T=\frac{\pi}{\frac{1}{\sqrt{2}}}=\sqrt{2}\pi.$$

Answer: $\pi\sqrt{2}.$

$f(t)=\cos^4 t+\sin t.$

Solution:

$$\cos^4 t=\left(\cos^2 t\right)^2=\left(\frac{1+\cos 2t}{2}\right)^2=\frac{1+2\cos 2t+\cos^2 2t}{4}=\frac{1+2\cos 2t+\frac{1+\cos^4 t}{2}}{4}=$$

$$=\frac{2+4\cos 2t+1+\cos^4 t}{4}.$$

$$f(t)=\frac{3+4\cos 2t+\cos 4t}{4}+\sin t.$$

$$T_1=\frac{2\pi}{2}=\pi;\;\; T_2=\frac{2\pi}{4}=\frac{\pi}{2};\;\; T_3=\frac{2\pi}{1}=2\pi.$$

$$T=HCK\left(\pi;\frac{\pi}{2};2\pi\right)=2\pi.$$

Answer: $2\pi.$

Self-study assignments:

Find the period of a function:

$f(x)=\sin 8x.$ Answer: $\frac{\pi}{4}.$

$y=\cos 1{,}5x.$ Answer: $\frac{4\pi}{3}.$

$\varphi(x)=tg6x.$ Answer: $\frac{\pi}{6}.$

$\varphi(x)=ctg4x.$ Answer: $\frac{\pi}{4}.$

$f(x)=\frac{3x}{4}-\cos\frac{x}{3}.$ Answer: $24\pi.$

$y=\sin\frac{2\pi}{3}.$ Answer: $3.$

$y=ctg\left(\frac{x}{\sqrt{2}}-\frac{\pi}{4}\right).$ Answer: $\pi\sqrt{2}.$

$y=\sin x\cdot\cos x.$ Answer: $\pi.$

$y=\sin 3x-\sin\frac{x}{3}.$ Answer: $6\pi.$

$y=\cos\frac{\pi x}{3}+\cos\frac{\pi x}{2}.$ Answer: $12.$

$y=\sin^4 x+\cos^4 x.$ Answer: $\frac{\pi}{2}.$

$f(x)=\sin 2x+3\sin(3x-2)-0{,}5\cos\left(\frac{4}{5}x+1\right).$ Answer: $10\pi.$