Section 12
Systems of equations

Two or more equations with *n* variables form a system when you need to find their general solutions.
The solution of a system of equations with *n* variables is such an ordered *n* numbers, which transforms each equation into the correct numerical equality.
To solve a system of equations means to find the set of all its solutions or to show that it has no solutions.
A system that has solutions is called joint, and not having solutions is called incompatible or contradictory.
The fact that the given system is written using curly brackets written to the left of the column of equations.
System of equations *(A)* are called a consequence of the system *(B)*, if all system solutions *(B)* are solutions of the system *(A)*.
This fact is written like this: $B \Rightarrow A$.
Two systems of equations called tantamount or equivalent if each of them is a consequence of the other. $B \Leftrightarrow A$.
Two systems are said to be equivalent if they both have no solutions.
There are three traditional ways to solve systems of two equations:

1) substitution method;
2) algebraic addition method;
3) comparison method.

The essence of this method lies in the fact that the systems express the same variable from both equations, for example *y* through *x*. Form a third equation in the variable *x* and solve it. Find the value of another variable. Write in response.

$$\begin{cases} x^2 - 3xy + y^2 + 2x + 3y = 6, (1) \\ 2x - y = 3. (2) \end{cases}$$

Solution:

Let's solve this system by substitution. From equation (2) we have:
$y = 2x - 3$.
Substitute the values into the equation (1):

$$x^2 - 3x(2x-3) + (2x-3)^2 + 2x + 3(2x-3) = 6,$$
$$x^2 - 6x^2 + 9x + 4x^2 - 12x + 9 + 2x + 6x - 9 = 6,$$
$$-x^2 + 5x - 6 = 0 \,|\cdot(-1)$$

$x^2 - 5x + 6 = 0$. By Vieta's theorem: $\left[\begin{array}{l} x_1 = 2,\ y_1 = 2\cdot 2 - 3 = 1 \\ x_2 = 3\ \ y_2 = 2\cdot 3 - 3 = 3 \end{array}\right.$

Answer: $(2;1)$, $(3;3)$.

$$\begin{cases} x - y = 8, (1) \\ x \cdot y = -15\,(2) \end{cases}$$

Solution:

From equation (1), we express the variable y through x: $y = x - 8$;
Substitute the value of y in equation (2):
$x \cdot (x-8) = -15,\ -8x + x^2 + 15 = 0,\ x^2 - 8x + 15 = 0.$

$$\left[\begin{array}{l} x_1 = 3,\ \ y_1 = 3-8 = -5, \\ x_2 = 5\ \ \ y_2 = 5-8 = -3 \end{array}\right. \left[\begin{array}{l} y_1 = -5, \\ y_2 = -3. \end{array}\right.$$

Answer: $(3;-5),\ (5;-3).$

$$\begin{cases} x^2 + y^2 = 34, (1) \\ x - y = 2\,(2) \end{cases}$$

Solution:

Let us square both sides of equation (2):
$(x-y)^2 = 2^2,\ \ x^2 - 2xy + y^2 = 4\ \ (3).$
Subtract equation (1) from (3):

$$\begin{array}{r} x^2 - 2xy + y^2 = 4, \\ - \quad x^2 + y^2 = 34 \\ \hline -2xy = -30 : (-2) \\ xy = 15\,(4). \end{array}$$

Consider the system of equations (2): (4):

$$\begin{cases} x - y = 2, \\ xy = 15 \end{cases} \begin{cases} y = x - 2, \\ x(x-2) = 15 \end{cases} \begin{cases} y = x - 2, \\ x^2 - 2x - 15 = 0 \end{cases} \begin{cases} y_1 = -5;\ y_2 = 3, \\ x_1 = -3;\ x_2 = 5. \end{cases}$$

Answer: $-(3;-5),\ (5;3).$

$$\begin{cases} x^2 - y^2 = 40,\ (1) \\ xy = 21\ (2) \end{cases}$$

Solution:

Raise both sides of the equation (2) in the square:

$$\begin{cases} x^2 - y^2 = 40, \\ x^2 y^2 = 441. \end{cases} \begin{cases} x^2 + (-y^2) = 40, \\ x^2 \cdot (-y^2) = 441. \end{cases}$$

Based on Vieta's theorem, we compose an auxiliary equation:
$t^2 - 40t - 441 = 0.$ By Vieta's theorem: $t_1 = -9,\ t_2 = 49.$

$$\left[\begin{array}{l} \begin{cases} x^2 = -9, \\ -y^2 = 49. \end{cases} \\ \begin{cases} x^2 = 49, \\ -y^2 = -9. \end{cases} \end{array}\right. \left[\begin{array}{l} \begin{cases} x^2 = -9, \\ y^2 = -49. \end{cases} \\ \begin{cases} x^2 = 49, \\ y^2 = 9. \end{cases} \end{array}\right. \left[\begin{array}{l} x \in \varnothing \\ x \in \varnothing \\ x_1 = -7 \\ x_2 = 7 \\ y_1 = -3 \\ y_2 = 3 \end{array}\right. \left[\begin{array}{l} x_1 = -7 \\ y_1 = -3 \end{array}\right. \left[\begin{array}{l} x_2 = 7 \\ y_2 = 3 \end{array}\right.$$

Answer: $(-7;\ -3),\ (7;\ 3).$

$$\begin{cases} x^2 + y^2 = 80,\ (1) \\ xy = 32\ (2) \end{cases}$$

Solution:

Multiply equation (2) by 2: $2xy = 64\ \ (3).$

Add the equations (1) and (3):

$$+\begin{cases} x^2+y^2=80, \\ 2xy=64 \end{cases}$$

$$x^2+2xy+y^2=144;$$

$$(x+y)^2=144;$$

$$\sqrt{(x+y)^2}=\sqrt{12^2}.$$

$$|x+y|=|12|;$$

Because the $12>0$, то $|12|=12$.

$|x+y|=12.$ *(4)*

Subtract the equation (1)–(3):

$$-\begin{cases} x^2+y^2=80, \\ 2xy=64 \end{cases}$$

$$x^2-2xy+y^2=144;$$

$$(x-y)^2=4^2;$$

$$\sqrt{(x-y)^2}=\sqrt{4^2}.$$

$$|x-y|=|4|;$$

Because the $4>0$, то $|4|=4$.

$|x-y|=4$ *(5).*

From equations (4) and (5) we form the system:

$\begin{cases} |x+y|=12, \\ |x-y|=4. \end{cases}$ By the properties of the modulus of a number, we have:

$\begin{cases} x+y=\mu 12, \\ x-y=\mu 4. \end{cases}$ With this combination will create a set of four systems of equations:

$$\left[\begin{array}{llll}
\begin{cases} x+y=-12, \\ x-y=-4. \end{cases} & +\begin{cases} x+y=-12 \\ x-y=-4 \end{cases} \quad 2x=-16; x=-8 & \begin{cases} -8+y=-12 \\ x=-8 \end{cases} & \left[\begin{array}{l} y=-4, \\ x=-8. \end{array}\right. \\
\begin{cases} x+y=-12, \\ x-y=4. \end{cases} & +\begin{cases} x+y=-12 \\ x-y=4 \end{cases} \quad 2x=-8; x=-4 & \begin{cases} -4+y=12 \\ x=-4 \end{cases} & \left[\begin{array}{l} y=-8, \\ x=-4. \end{array}\right. \\
\begin{cases} x+y=12, \\ x-y=-4. \end{cases} & +\begin{cases} x+y=12 \\ x-y=-4 \end{cases} \quad 2x=8; x=4 & \begin{cases} 4+y=12 \\ x=4 \end{cases} & \left[\begin{array}{l} y=8, \\ x=4. \end{array}\right. \\
\begin{cases} x+y=12, \\ x-y=4. \end{cases} & +\begin{cases} x+y=12 \\ x-y=4 \end{cases} \quad 2x=16; x=8 & \begin{cases} 8+y=12 \\ x=8 \end{cases} & \left[\begin{array}{l} y=4, \\ x=8. \end{array}\right.
\end{array}\right.$$

Answer: $(-4;-8),(-8;-4),(4;8),(8;4).$

$$2x^2-15xy+4y^2-12x+45y-24=0, \ (1)$$

$$x^2+xy-2y^2-3x+3y=0 \ (2)$$

Solution:

We write equation (2) as square with respect to *x:*

$$x^2+(xy-3x)+(3y-2y^2)=0; \ x^2+(y-3)x+3y-2y^2=0; (3)$$

$$D=(y-3)^2-4\cdot(3y-2y^2)=y^2-6y+9-12y+8y^2=9y^2-18y+9=$$

$$=9\cdot(y^2-2y+1)=3^2\cdot(y-1)^2=(3\cdot(y-1))^2.$$

$$x_1=\frac{-(y-3)-\sqrt{(3\cdot(y-1))^2}}{2}=\frac{-y+3-3\cdot(y-1)}{2}=\frac{-y+3-3y+3}{2}=\frac{6-4y}{2}=3-2y;$$

$$x_2 = \frac{-y+3+3y-3}{2} = \frac{2y}{2} = y.$$

According to the formula of decomposition quadratic polynomial factoring $ax^2 + bx + c = a \cdot (x - x_1) \cdot (x - x_2)$ factorize the left side of the equation *(3):*

$x^2 + (y-3)x + 3y - 2y^2 = (x-y)\cdot(x-3+2y);$

$(x-y)\cdot(x-3+2y) = 0.$ This equation is equivalent to two equations:

$x - y = 0$ і $x + 2y - 3 = 0.$

The original system of equations is equivalent to the set of two equations:

$$\left[\begin{array}{l} \begin{cases} 2x^2 - 15xy + 4y^2 - 12x + 45y - 24 = 0, \\ x - y = 0. \end{cases} \\ \begin{cases} 2x^2 - 15xy + 4y^2 - 12x + 45y - 24 = 0, \\ x + 2y - 3 = 0. \end{cases} \end{array}\right.$$

We solve each system of equations of the set by the method of substitution:

$$\left[\begin{array}{l} \begin{cases} y = x, \\ 2x^2 - 15x^2 + 4x^2 - 12x + 45x - 24 = 0. \end{cases} \\ \begin{cases} x = 3 - 2y, \\ 2\cdot(3-2y)^2 - 15\cdot(3-2y)\cdot y + 4y^2 - 12\cdot(3-2y) + 45y - 24 = 0. \end{cases} \end{array}\right.$$

$$\left[\begin{array}{l} \begin{cases} y = x, \\ -9x^2 + 33x - 24 = 0 \,|: (-3). \end{cases} \\ \begin{cases} x = 3 - 2y, \\ 2\cdot(9 - 12y + 4y^2) - 45y + 30y^2 + 4y^2 - 36 + 24y + 45y - 24 = 0. \end{cases} \end{array}\right.$$

$$\left[\begin{array}{l} \begin{cases} y = x, \\ 3x^2 + 11x + 8 = 0. \end{cases} \\ \begin{cases} x = 3 - 2y, \\ 18 - 24y + 8y^2 - 45y + 30y^2 + 4y^2 - 36 + 24y + 45y - 24 = 0. \end{cases} \end{array}\right.$$

$$\left[\begin{array}{l} \begin{cases} y = x, \\ 3x^2 - 11x + 8 = 0 \end{cases} \quad D = 121 - 96 = 25, \left[\begin{array}{l} y_1 = 1, \\ x_1 = \dfrac{11-5}{6} = 1 \end{array}\right. \left[\begin{array}{l} y_2 = \dfrac{8}{3}, \\ x_2 = \dfrac{16}{6} = \dfrac{8}{3}. \end{array}\right. \\ \begin{cases} x = 3 - 2y, \\ 42y^2 - 42 = 0 \end{cases} \left[\begin{array}{l} x_3 = 5, \\ y_3 = -1 \end{array}\right. \left[\begin{array}{l} x_4 = 1, \\ y_4 = 1. \end{array}\right. \end{array}\right.$$

Answer: $(1;1), \left(\frac{8}{3}; \frac{8}{3}\right), (5;-1).$

$$\begin{cases} x^3 = 5x + y, \ (1) \\ y^3 = x + 5y. \ (2) \end{cases}$$

Solution:

Add equations (1) and (2):

$x^3 + y^3 = 6x + 6y;$

$x^3 + y^3 = 6 \cdot (x + y)$ (3)

Subtract equations (1) and (2):

$x^3 - y^3 = 4x + 4y;$

$x^3 - y^3 = 4 \cdot (x - y)$ (4).

Consider the system of equations (3) and (4).

Let us apply to its solution the formulas for the sum and difference of cubes:

$$\begin{cases} x^3 + y^3 = 6 \cdot (x + y), \\ x^3 - y^3 = 4 \cdot (x - y) \end{cases} \begin{cases} (x + y) \cdot (x^2 - xy + y^2) - 6 \cdot (x + y) = 0, \\ (x - y) \cdot (x^2 + xy + y^2) - 4 \cdot (x - y) = 0. \end{cases}$$

$$\begin{cases} (x + y) \cdot (x^2 - xy + y^2 - 6) = 0, \\ (x - y) \cdot (x^2 + xy + y^2 - 4) = 0. \end{cases}$$

This system of equations is equivalent to a set of four systems of equations:

$$\left[\begin{array}{l} \begin{cases} x + y = 0, \\ x - y = 0. \end{cases} \begin{cases} 2x = 0, \\ x - y = 0. \end{cases} \left[\begin{array}{l} x_1 = 0, \\ y_1 = 0. \end{array} \right. \\ \begin{cases} x - y = 0, \\ x^2 - xy + y^2 - 6 = 0. \end{cases} \begin{cases} x = y, \\ x^2 - x^2 + y^2 - 6 = 0. \end{cases} \begin{cases} x = y, \\ y^2 - 6 = 0. \end{cases} \begin{cases} x_2 = -\sqrt{6}, \\ y_2 = -\sqrt{6}. \end{cases} \left[\begin{array}{l} x_3 = \sqrt{6}, \\ y_3 = \sqrt{6}. \end{array} \right. \\ \begin{cases} x + y = 0, \\ x^2 + xy + y^2 - 4 = 0. \end{cases} \begin{cases} x = -y, \\ (-y)^2 - y^2 + y^2 - 4 = 0. \end{cases} \begin{cases} x = -y, \\ y^2 = 4. \end{cases} \left[\begin{array}{l} x_4 = 2, \\ y_4 = -2, \end{array} \right. \left[\begin{array}{l} x_5 = -2, \\ y_5 = 2. \end{array} \right. \\ + \begin{cases} x^2 - xy + y^2 - 6 = 0, \\ x^2 + xy + y^2 - 4 = 0 \end{cases} \qquad - \begin{cases} x^2 - xy + y^2 - 6 = 0, \\ x^2 + xy + y^2 - 4 = 0 \end{cases} \end{array} \right.$$

$$2x^2 + 2y^2 - 10 = 0 \qquad -2xy - 2 = 0,$$

$$x^2 + y^2 = 5 \ (5) \qquad 2xy = -2 \ (6)$$

Let us form a system of equations (5) and (6):

$$\begin{cases} x^2 + y^2 = 5, \\ 2xy = -2. \end{cases}$$

We add and subtract the equation of this system and form a new system of equations:

$$\begin{cases} x^2 + 2xy + y^2 = 3, \\ x^2 - 2xy + y^2 = 4. \end{cases} \begin{cases} (x + y)^2 - 3 = 0, \\ (x - y)^2 - 7 = 0. \end{cases} \begin{cases} (x + y - \sqrt{3}) \cdot (x + y + \sqrt{3}) = 0, \\ (x - y - \sqrt{7}) \cdot (x - y + \sqrt{7}) = 0. \end{cases}$$

The latter system is equivalent to the set of four systems of equations:

$$\left[\begin{array}{l}\begin{cases}x+y-\sqrt{3}=0,\\ x+y-\sqrt{7}=0.\end{cases}\\ \begin{cases}x+y-\sqrt{3}=0,\\ x-y+\sqrt{7}=0.\end{cases}\\ \begin{cases}x+y+\sqrt{3}=0,\\ x-y-\sqrt{7}=0.\end{cases}\\ \begin{cases}x+y+\sqrt{3}=0,\\ x-y+\sqrt{7}=0.\end{cases}\end{array}\right. \quad \left[\begin{array}{l}x_6=\frac{1}{2}\cdot\left(\sqrt{3}+\sqrt{7}\right),\\ y_6=\frac{1}{2}\cdot\left(\sqrt{3}-\sqrt{7}\right)\end{array}\right. \quad \left[\begin{array}{l}x_7=\frac{1}{2}\cdot\left(\sqrt{3}-\sqrt{7}\right),\\ y_7=\frac{1}{2}\cdot\left(\sqrt{7}-\sqrt{3}\right)\end{array}\right. \quad \left[\begin{array}{l}x_8=\frac{1}{2}\cdot\left(\sqrt{7}-\sqrt{3}\right),\\ y_8=\frac{1}{2}\cdot\left(\sqrt{7}+\sqrt{3}\right)\end{array}\right. \quad \left[\begin{array}{l}x_9=-\frac{1}{2}\cdot\left(\sqrt{3}+\sqrt{7}\right),\\ y_9=\frac{1}{2}\cdot\left(\sqrt{7}-\sqrt{3}\right).\end{array}\right.$$

Answer: $(0;\ 0),\ \left(-\sqrt{6};\ -\sqrt{6}\right),\ \left(\sqrt{6};\ \sqrt{6}\right),\ (2;\ -2),\ (-2;\ 2),\ \left(\frac{\sqrt{3}+\sqrt{7}}{2};\ \frac{\sqrt{3}-\sqrt{7}}{2}\right),$ $\left(\frac{\sqrt{3}-\sqrt{7}}{2};\ \frac{\sqrt{7}-\sqrt{3}}{2}\right),\ \left(\frac{\sqrt{7}-\sqrt{3}}{2};\ \frac{\sqrt{7}+\sqrt{3}}{2}\right),\ \left(-\frac{\sqrt{3}+\sqrt{7}}{2};\ \frac{\sqrt{7}-\sqrt{3}}{2}\right).$

$$\begin{cases}3x^2+2xy+y^2=11,\\ x^2+2xy+3y^2=17.\end{cases}$$

Solution:

We introduce a new variable $x=t\cdot y.$ Then the system is reduced to the following:

$$\begin{cases}3\cdot(ty)^2+2y(ty)+y^2=11,\\ (ty)^2+2y(ty)+3y^2=17.\end{cases}\Leftrightarrow\begin{cases}3t^2y^2+2ty^2+y^2=11,\\ t^2y^2+2ty^2+3y^2=17.\end{cases}\ \begin{cases}y^2\cdot\left(3t^2+2t+1\right)=11,\\ y^2\cdot\left(t^2+2t+3\right)=17.\end{cases}$$

Because the $y\neq 0,$ then we separate the left and right sides of the equations of the last system:

$$\frac{y^2\cdot\left(3t^2+2t+1\right)}{y^2\cdot\left(t^2+2t+3\right)}=\frac{11}{17}\Rightarrow\frac{3t^2+2t+1}{t^2+2t+3}=\frac{11}{17};$$

Applying the main property of proportion, we get:

$$17\cdot\left(3t^2+2t+1\right)=11\cdot\left(t^2+2t+3\right)\Rightarrow 51t^2+34t+17=11t^2+22t+33;$$

$$40t^2+12t-16=0\,\big|:4\ \Rightarrow 10t^2+3t-4=0.\ D=9+160=169,$$

$$t_1=\frac{-3-13}{20}=-\frac{16}{20}=-\frac{4}{5};\ t_2=\frac{-3+13}{20}=\frac{10}{20}=\frac{1}{2}.$$

Substitute the value $t_1=-\frac{4}{5}$ into the equation $y^2\cdot\left(t^2+2t+3\right)=17.$

$$y^2\cdot\left(\left(-\frac{4}{5}\right)^2+2\cdot\left(-\frac{4}{5}\right)+3\right)=17,\ \Rightarrow y^2\cdot\left(\frac{16}{25}-\frac{8}{5}+3\right)=17,\ y^2=\frac{16-40+75}{25}=17;$$

$$y^2\cdot\frac{51}{25}=17;\ y^2=17:\frac{51}{25};\ y^2=\frac{17}{1}\cdot\frac{25}{51}=\frac{25}{3};\ y_1=-\frac{5}{\sqrt{3}}=-\frac{5\sqrt{3}}{3};\ y_2=\frac{5\sqrt{3}}{3}.$$

Back to replacement $x = ty$:

$$x_1 = -\frac{4}{5}\cdot\left(-\frac{5\sqrt{3}}{3}\right) = \frac{4\sqrt{3}}{3};\ x_2 = -\frac{4}{5}\cdot\frac{5\sqrt{3}}{3} = -\frac{4\sqrt{3}}{3}.$$

Substituting the value of $t_2 = \frac{1}{2}$ into the equation $y^2\cdot\left(3t^2+2t+1\right)=11,$ we get:

$$y^2\cdot\left(3\cdot\left(\frac{1}{2}\right)^2+2\cdot\frac{1}{2}+1\right)=11,\ \ y^2\cdot\left(\frac{3}{4}+2\right)=11;\ y^2=11:2\frac{3}{4}=\frac{11}{1}\cdot\frac{4}{11}=4;$$

$y_3 = -2;\ y_3 = 2.$

$x_3 = -2\cdot\frac{1}{2} = -1;\ x_4 = \frac{1}{2}\cdot 2 = 1.$

Answer: $\left(\frac{4\sqrt{3}}{3};\ -\frac{5}{3}\sqrt{3}\right),\ \left(-\frac{4\sqrt{3}}{3};\ \frac{5}{3}\sqrt{3}\right),\ (-1;-2),(1;2).$

$$\begin{cases} x^2+y^2+x+y=8, \\ x^2+y^2+xy=7. \end{cases}$$

Solution:

Let's replace $x+y=u,$ and $x\cdot y=v.$

$(x+y)^2=u^2,\ x^2+2xy+y^2=u^2,\ x^2+y^2=u^2-2v.$

The original system takes the form:

$$\begin{cases} u^2-2v+u=8, \\ u^2-2v+v=7 \end{cases}\Rightarrow\begin{cases} u^2+u-2v=8, \\ u^2-v=7. \end{cases}$$

We solve this system by the substitution method.

$v=u^2-7;\ u^2+u-2\left(u^2-7\right)=8,\ u^2+u-2u^2+14=8,\ -u^2+u+6=0\big|\cdot(-1),$

$u^2-u-6=0.$

By Vieta's theorem, we have: $u_1=3,\ u_2=-2.$ Then $v_1=3^2-7=2;\ v_2=(-2)^2-7=-3.$

Let's solve a set of systems of equations:

$$\left[\begin{array}{l}\begin{cases} x+y=-2, \\ xy=-3. \end{cases} \\ \begin{cases} x+y=3, \\ xy=2. \end{cases}\end{array}\right.\ \left[\begin{array}{l}\begin{cases} y=-2-x, \\ x\cdot(-2-x)=-3. \end{cases} \\ \begin{cases} y=3-x, \\ x\cdot(3-x)=2. \end{cases}\end{array}\right.\ \left[\begin{array}{l}\begin{cases} y=-2-x, \\ -2x-x^2+3=0. \end{cases} \\ \begin{cases} y=3-x, \\ 3x-x^2-2=0. \end{cases}\end{array}\right.\Rightarrow\left[\begin{array}{l}\begin{cases} y=-2-x, \\ x^2+2x-3=0. \end{cases} \\ \begin{cases} y=3-x, \\ x^2-3x+2=0. \end{cases}\end{array}\right.$$

$$\left[\begin{array}{l}\begin{cases} y=-2-x, \\ x_1=-3;\ x_2=1. \end{cases} \\ \begin{cases} y=3-x, \\ x_3=1;\ x_4=2. \end{cases}\end{array}\right.$$

Answer: $(-3;\ 1),\ (1;\ -3),\ (1;\ 2),\ (2;\ 1).$

Let us show some methods for solving systems that include irrational equations:

$$\begin{cases} x+\sqrt{\frac{x}{y}}=\frac{2}{y}, \\ y-x=3. \end{cases}$$

Solution:

Here it is advisable to look for the range of admissible values of the system:

$$\begin{cases}\dfrac{x}{y}\ge 0,\\ y\ne 0\end{cases}\Rightarrow\begin{matrix}\begin{cases}x\ge 0,\\ y>0.\end{cases} \text{– нехай це перша область ДЗН, а}\\ \begin{cases}x\le 0,\\ y<0.\end{cases} \text{це друга область ДЗН.}\end{matrix}$$

Let us solve this system in the first range of admissible values of the system:
We transform the first equation of the system by multiplying both sides of it by y here $(y>0)$:

$$x+\sqrt{\frac{x}{y}\cdot\frac{y}{y}}=\frac{2}{y}\Big|\cdot y,\ \ xy+\sqrt{xy}-2=0,\ \ \left(\sqrt{xy}\right)^2+\sqrt{xy}-2=0.$$

Let's introduce a new variable:

$\sqrt{xy}=t,\ \ t^2+t-2=0.$

By the theorem of Vieta:

$t_1=-2;\ \ t_2=1.$

$\sqrt{xy}=-2,\ \ \sqrt{xy}=-2,\ \ xy\in\varnothing.$

$\sqrt{xy}=1\Rightarrow xy=1.$

$$\begin{cases}\sqrt[3]{x}+\sqrt[3]{y}=2{,}5\sqrt[6]{xy},\ (1)\\ x-14y=100.\ (2)\end{cases}$$

Solution:

Range of valid values: $x>0,\ \ y>0.$ $\sqrt[3]{x}=\sqrt[6]{x^2},\ \ \sqrt[3]{x}=\sqrt[6]{y^2},\ \ \sqrt[6]{x^2}+\sqrt[6]{y^2}=2{,}5\sqrt[6]{xy}\Big|:\sqrt[6]{xy}.$

$\sqrt[6]{\dfrac{x^2}{xy}}+\sqrt[6]{\dfrac{y^2}{xy}}=2{,}5;\ \ \sqrt[6]{\dfrac{x}{y}}+\sqrt[6]{\dfrac{y}{x}}-2{,}5=0.$ We denote $\sqrt[6]{\dfrac{x}{y}}=t,$ then $t+\dfrac{1}{t}-2{,}5=0;$

$$\begin{cases}t^2-2{,}5t+1=0,\ D=6{,}25-4=2{,}25=1{,}5^2.\\ t\ne 0\end{cases}$$

$t_1=\dfrac{2{,}5-1{,}5}{2}=\dfrac{1}{2};\ t_2=\dfrac{2{,}5+1{,}5}{2}=2.\ \sqrt[6]{\dfrac{x}{y}}=\dfrac{1}{2};\ \dfrac{x}{y}=\left(\dfrac{1}{2}\right)^6;\ \dfrac{x}{y}=\dfrac{1}{64};\ \ y=64x\ \textit{(3)}.$

$\sqrt[6]{\dfrac{x}{y}}=2,\ \dfrac{x}{y}=2^6,\ x=64y\ \textit{(4)}.$

We form a set of two systems of equations:

$$\left[\begin{array}{l}\begin{cases}y=64x,\\ x-14y=100.\end{cases}\\ \begin{cases}x=64y,\\ x-14y=100.\end{cases}\end{array}\right.\ \left[\begin{array}{l}\begin{cases}y=64x,\\ x-14\cdot 64x=100.\end{cases}\\ \begin{cases}x=64y,\\ 64y-14y=100.\end{cases}\end{array}\right.\ \left[\begin{array}{l}\begin{cases}y=64x,\\ -895x=100.\end{cases}\\ \begin{cases}x=64y,\\ 50y=100.\end{cases}\end{array}\right.\ \begin{array}{l}\begin{cases}y=64\cdot x,\\ x=-\dfrac{100}{895}\end{cases}\notin \text{ОДЗН}\ \varnothing.\\ \begin{cases}x=64\cdot 2,\\ y=\dfrac{100}{50}=2.\end{cases}\ \begin{cases}x=128,>0\\ y=2.>0\end{cases}\end{array}$$

Answer: $(128;2)$.

$$\begin{cases}x+y-\sqrt{\dfrac{x+y}{x-y}}=\dfrac{6}{x-y},\ (1)\\ xy=20\ (2)\end{cases}$$

Solution:

We transform $\sqrt{\frac{x+y}{x-y}}=\sqrt{\frac{(x+y)\cdot(x-y)}{(x-y)^2}}=\frac{\sqrt{x^2-y^2}}{|x-y|}$, then equation (1) becomes:

$x+y-\frac{\sqrt{x^2-y^2}}{|x-y|}=\frac{6}{x-y}$, if $x-y>0$, то $|x-y|=x-y$.

$x+y-\frac{\sqrt{x^2-y^2}}{|x-y|}=\frac{6}{x-y}\Big|\times(x-y)$, $x^2-y^2-\sqrt{x^2-y^2}-6=0$. We denote $\sqrt{x^2-y^2}=t$.

$t^2-t-6=0$, $t_1=3$, $t_2=-2$.

$\sqrt{x^2-y^2}=3$; $\begin{cases} x^2-y^2=9, \\ xy=20 \end{cases}$ $\sqrt{x^2-y^2}=-2$ $\varnothing$ $y=\frac{20}{x}$;

$x^2-\left(\frac{20}{x}\right)^2=9$, $x^2-\frac{400}{x^2}-9=0$, $\begin{cases} x^4-9x^2-400=0, \\ x^2\neq 0 \end{cases}$ $x^2=m$, $m^2-9m-400=0$.

$D=81+4\cdot 400=1681=41^2$, $m_1=\frac{9-41}{2}=-\frac{32}{2}=-16$; $m_2=\frac{9+41}{2}=25$;

$x^2=-16$ $\quad x\in\varnothing$. $x^2=25$, $x_1=-5$; $x_2=5$; $y_1=\frac{20}{-5}=-4$; $y_2=\frac{20}{5}=4$.

$x-y>0$ $\quad -5-(-4)=-1<0$. Means, (5; 4).

If $x-y<0$, то $|x-y|=-(x-y)$. Then $x+y-\frac{\sqrt{x^2-y^2}}{-(x-y)}-\frac{6}{x-y}=0$;

$x^2-y^2+\sqrt{x^2-y^2}-6=0$; $\sqrt{x^2-y^2}=l$; $l^2+l-6=0$, $l_1=-3$; $l_2=2$.

$\sqrt{x^2-y^2}=-3$, $\varnothing$ $\sqrt{x^2+y^2}=2$, $x^2-y^2=4$

$\begin{cases} x^2-y^2=4, \\ xy=20. \end{cases}$ $y=\frac{20}{x}$, $x^2-\frac{400}{x^2}-4=0$, $\begin{cases} x^4-4x^2-400=0, \\ x^2\neq 0. \end{cases}$ $x^2=n$, $n^2-4n-400=0$,

$D=16+1600=1616$, $n_1=\frac{4-\sqrt{1616}}{2}=0$, $n_2=\frac{4+\sqrt{1616}}{2}$; $x^2=\frac{4+\sqrt{1616}}{2}$,

$x_1=\sqrt{\frac{4+\sqrt{1616}}{2}}=\sqrt{2+2\sqrt{101}}$; $x_2=\sqrt{2+2\sqrt{101}}$; $y_2=\frac{20}{\sqrt{2+2\sqrt{101}}}$.

Answer: (5; 4), $\left(-\sqrt{2+2\sqrt{101}};\ \frac{20}{-\sqrt{2+2\sqrt{101}}}\right)$.

A system of equations was formed:

$\begin{cases} y-x=3, \\ xy=1. \end{cases}$ $\begin{cases} y=3+x, \\ x\cdot(3+x)=1. \end{cases}$ $\begin{cases} y=3+x, \\ 3x+x^2-1=0. \end{cases}$ $\begin{cases} y=3+x, \\ x^2+3x-1=0. \end{cases}$

$D=9+4=13>0$; $x_1=\frac{-3-\sqrt{13}}{2}$; $x_2=\frac{-3+\sqrt{13}}{2}$.

$y_1=3+\frac{-3-\sqrt{13}}{2}=\frac{3-\sqrt{13}}{2}$; $y_2=3+\frac{-3+\sqrt{13}}{2}=\frac{3+\sqrt{13}}{2}$.

As $\frac{-3-\sqrt{13}}{2}$ i $\frac{3-\sqrt{13}}{2}$ do not belong to the first range of valid values, in which this system is solved, and $\frac{-3+\sqrt{13}}{2}$ and $\frac{3+\sqrt{13}}{2}$ belonging to it, then the solution to the system is a pair of numbers $\left(\frac{-3+\sqrt{13}}{2};\ \frac{3+\sqrt{13}}{2}\right)$.

Let us simplify the first equation of the system into the second range of valid values:

$x+\sqrt{\frac{x\cdot y}{y\cdot y}}=\frac{2}{y}\Big|\cdot y,\ \ xy+\sqrt{xy}-2=0,$ а тому $\left(\sqrt{xy}\right)^2-\sqrt{xy}-2=0,\ \ \sqrt{xy}=t,\ \ t^2-t-=0,$

$t_1=-1;\ t_2=2.\ \ \sqrt{xy}=-1,\ \ xy\in\varnothing.\ \ \sqrt{xy}=2,\ \ xy=4.$ We have such a system:

$$\begin{cases}y-x=3,\\ xy=4\end{cases}\to\begin{cases}y=3+x,\\ x\cdot(3+x)=4\end{cases}\to\begin{cases}y=3+x,\\ 3x+x^2=4\end{cases}\to\begin{cases}y=3+x,\\ x^2+3x-4=0\end{cases}$$

$$\begin{cases}y=3+x,\\ x=-4;\ x=1\end{cases}\ \begin{cases}x=-4,\\ y=3-4=-1\end{cases}\ \begin{cases}x=1,\\ y=3+1=4.\end{cases}$$

Pair (1; 4) does not belong to the second range of valid values, and therefore the solution to the system is the pair (-1; -4).

Answer: $\left(\frac{-3+\sqrt{13}}{2};\ \frac{3+\sqrt{13}}{2}\right),\ (-4;\ -1).$

$$\begin{cases}3|x|+5y+9=0,\\ 2x-|y|-7=0.\end{cases}$$

Solution:

Let's solve this system in each of the four range of valid values:

I). $\begin{cases}x\ge 0,\\ y\ge 0.\end{cases}\ \ \begin{matrix}|x|=x.\\ |y|=y.\end{matrix}\ \ \begin{cases}3x+5y+9=0,\\ 2x-y-7=0\,\big|\cdot 5\end{cases}\ \ \begin{cases}3x+5y=-9,\\ 10x-5y=35\end{cases}$

$$13x=26;$$

$$x=2$$

$3\cdot 2+5y=-9, 8y=-9-6,\ \ y=-3$ does not satisfy the condition $y\ge 0.$

In this range of valid values the system has no solutions.

$$\begin{cases}x^2-x\sqrt{xy}=8,\\ y^2-y\sqrt{xy}=-1.\end{cases}$$

Solution:

Replacement $x=ty,\ \ y\ne 0.$

$$\begin{cases}(ty)^2-ty\sqrt{ty\cdot y}=8,\\ y^2-y\sqrt{ty\cdot y}=-1\end{cases}\ \begin{cases}t^2y^2-ty\sqrt{ty^2}=8,\\ y^2-y\sqrt{ty^2}=-1\end{cases}\ \begin{cases}t^2y^2-ty|y|\sqrt{t}=8,\\ y^2-y|y|\sqrt{t}=-1.\end{cases}$$

If $y>0,$ then $|y|=y$

$$\begin{cases}2y^2-ty^2\sqrt{t}=8,\\ y^2-y^2\sqrt{t}=-1.\end{cases}\ t^2y^2-ty^2\sqrt{t}=8,\ y^2\cdot\left(t^2-\sqrt{t}\right)=8,\ \frac{t^2-t\sqrt{t}}{1-\sqrt{t}}=-8,\ \frac{t\cdot\left(t-\sqrt{t}\right)}{1-\sqrt{t}}=-8,$$
$$y^2-y^2\sqrt{t}=-1.\ \ y^2\left(1-\sqrt{t}\right)=-1.$$

$\frac{t\cdot\sqrt{t}\left(\sqrt{t}-1\right)}{1-\sqrt{t}}=-8,\ \frac{t\sqrt{t}\left(1-\sqrt{t}\right)}{1-\sqrt{t}}=8,\ t\sqrt{t}=8,\ t=2^2=4.\quad t^2=2.$ Exponentiation $\frac{2}{3}$:

$\begin{cases}x=4y,\\ y^2-y\sqrt{xy}=-1\end{cases}\ \begin{cases}x=4y,\\ y^2-y\sqrt{4y\cdot y}+1=0\end{cases}\ \begin{cases}-4y=0,\\ y^2-2y^2+1=0\end{cases}\ \begin{cases}x-4y=0,\\ 1-y^2=0.\end{cases}\ \begin{cases}x=4\cdot 1,\\ y=1\end{cases}\ \begin{matrix}x=4,\\ y=1.\end{matrix}$

If $y<0,$ then $|y|=-y$ i $\begin{cases}t^2y^2+ty^2\sqrt{t}=8,\\ y^2+y^2\sqrt{t}=-1\end{cases}\ \frac{y^2\cdot\left(t^2+t\sqrt{t}\right)}{y^2\cdot\left(1+\sqrt{t}\right)}=\frac{8}{1},\ \frac{t\sqrt{t}\cdot\left(1+\sqrt{t}\right)}{1+\sqrt{t}}=-8,$

$t\sqrt{t}=-8,\ t\in\varnothing.$

Answer: *(4; 1)*.

II). $\begin{cases}x\ge 0,\\ y<0\end{cases}\ \begin{cases}|x|=x,\\ |y|=-y.\end{cases}\ \begin{cases}3x+5y=-9,\\ 2x+y=7\,|\cdot(-5)\end{cases}\ +\begin{cases}3x+5y=-9,\\ -10x-5y=-35\end{cases}$

$-7x=-44;$

$x=\frac{-44}{-7}=\frac{44}{7}\ge 0;\ 3\cdot\frac{44}{7}+5y=-9;\ 5y=-9-\frac{132}{7}=\frac{-195}{7};\ y=-\frac{195}{7}:5=-\frac{195}{7{,}5}=-\frac{39}{7}<0.$

$\left(\frac{44}{7};\ -\frac{39}{7}\right)-$ Solving a system of equations.

III). $\begin{cases}x\le 0,\\ y\ge 0\end{cases}\ \begin{cases}|x|=-x,\\ |y|=y.\end{cases}\ \begin{cases}-3x+5y=-9,\\ 2x-y-7=0\,|\cdot 5\end{cases}\ +\begin{cases}-3x+5y=-9,\\ 10x-5y=35\end{cases}$

$7x=26;$

$x=\frac{26}{7}-$ does not satisfy the condition $x\le 0,$ and therefore the system in this region of admissible values of solutions does not have.

IV). $\begin{cases}x<0,\\ y<0\end{cases}\ \begin{cases}|x|=-x,\\ |y|=-y.\end{cases}\ \begin{cases}-3x+5y=-9,\\ 2x+y=7\,|\cdot(-5)\end{cases}\ +\begin{cases}-3x+5y=-9,\\ -10x-5y=-35\end{cases}$

$-13x=26;$

$x=\frac{26}{-13}=-2\in$ range of valid values. $2\cdot 2+y=7;\ y=3\notin$ range of valid values.

Answer: $\left(\frac{44}{7};\ -\frac{39}{7}\right)$.

$$\begin{cases}(x-y)\cdot 0{,}5^{y-x}=5\cdot 2^{x-y},\ (1)\\ (x-y)^{\frac{x+y}{7}}=125\ (2)\end{cases}$$

Solution:

We transform the first equation of the system:

$0{,}5^{y-x}=\left(\frac{1}{2}\right)^{y-x}=\left(2^{-1}\right)^{y-x}=2^{x-y},\ (x-y)\cdot 2^{x-y}=5\cdot 2^{x-y}\,\big|:2^{x-y},\ x-y=5.$

Substitute this value into the equation (2):

$5^{\frac{x+y}{7}}=125;\ 5^{\frac{x+y}{7}}=5^3.$ Since the exponential function is monotonic, we have:

$\frac{x+y}{7}=3\rightarrow x+y=21$ We form a new system of equations:

$\begin{cases} x+y=21. \\ x-y=5. \end{cases}$ $2x=26,$ $x=13.$ $13+y=21,$ $y=8.$

Answer: $(13;\ 8)$.

$$\begin{cases} (x+y)^{\frac{1}{x}}=2, \ (1) \\ (x+y)\cdot 4^x=64 \ (2) \end{cases}$$

Solution:

Raise the equation (1) to the power x:

$\left((x+y)^{\frac{1}{x}}\right)^x=2^x,$ $x+y=2^x.$ Then the equation (2) will look like $2^x\cdot 4^x=4^3;$

$2^x\cdot 2^{2x}=\left(2^2\right)^3,$ $2^{3x}=2^6,$ $3x=6,$ $x=2,$ $y=2^2-2=2.$

Answer: $(2;\ 2)$.

$$\begin{cases} 3^{2x-1}\cdot 27^{x+y}=3, \\ (5x-y)^2=36. \end{cases}$$

Solution:

$$\begin{cases} 3^{2x-1}\cdot 27^{x+y}=3, \\ \sqrt{(5x-y)^2}=\sqrt{36}. \end{cases} \quad \begin{cases} 3^{2x-1+3x+3y}=3^1, \\ |5x-y|=6. \end{cases}$$

$$\left[\begin{array}{l} \begin{cases} 5x+3y=2, \\ 5x-y=6 \,|\cdot 3 \end{cases} \\ \begin{cases} 5x+3y=2, \\ 5x-y=-6 \,|\cdot 3 \end{cases} \end{array}\right. \quad \left[\begin{array}{l} \begin{cases} 5x+3y=2, \\ 15x-3y=18 \end{cases} \\ \begin{cases} 5x+3y=2, \\ 15x-3y=-18 \end{cases} \end{array}\right. \quad \begin{array}{l} \begin{cases} 20x=20, \\ 5x+3y=2 \end{cases} \\ \begin{cases} 20x=-16, \\ 5x+3y=2. \end{cases} \end{array} \quad \left[\begin{array}{l} \begin{cases} x=1, \\ y=\dfrac{2-5\cdot 1}{3} \end{cases} \\ \begin{cases} x=-0{,}8, \\ y=\dfrac{2+5\cdot 0{,}8}{3} \end{cases} \end{array}\right. \quad \begin{array}{l} \left[\begin{array}{l} x=1, \\ y=-1 \end{array}\right. \\ \left[\begin{array}{l} x=-0{,}8, \\ y=2. \end{array}\right. \end{array}$$

Answer: $(1;-1),(-0{,}8;2)$.

Systems of logarithmic equations are solved in the same ways as systems of algebraic equations.

$$\begin{cases} x^{\lg y}=100, \ (1) \\ \log_y x=2 \ (2) \end{cases}$$

Solution:

$x>0, y>0, y\neq 1.$

Let us logarithm Eq. (1) with base 10:

$\lg y\cdot\lg x=\lg 100,$ $\lg y\cdot\lg x=2.$

By the definition of the logarithm of a number from equation (2), we have:

$x=y^2.$

Let's solve the system of equations:

$$\begin{cases} x=y^2, \\ \lg y\cdot\lg x=2 \end{cases} \quad \begin{cases} x=y^2, \\ \lg y\cdot\lg y^2=2 \end{cases} \quad \begin{cases} x=y^2, \\ 2\lg^2 y=2 \end{cases} \quad \begin{cases} x=y^2, \\ \lg^2 y=1 \end{cases}$$

This system of equations is equivalent to such a set of equations:

$$\left[\begin{array}{l}\begin{cases}x=y^2,\\ \lg y=-1;\end{cases}\\ \begin{cases}x=y^2,\\ \lg y=1;\end{cases}\end{array}\right. \quad \left[\begin{array}{l}\begin{cases}x=y^2,\\ y_1=10^{-1};\end{cases}\\ \begin{cases}x_2=100,\\ y_2=10;\end{cases}\end{array}\right.$$

Answer: $(0{,}01;0{,}1),(100;10)$.

$$\begin{cases}\sqrt{x+y}+\sqrt[3]{x-y}=6, & (1)\\ \sqrt[6]{(x+y)^3\cdot(x-y)^2}=8. & (2)\end{cases}$$

Solution:

We transform the second equation of the system:

$$\sqrt[6]{(x+y)^3\cdot(x-y)^2}=\sqrt[6]{(x+y)^3}\cdot\sqrt[6]{(x-y)^2}=\sqrt{x+y}\cdot\sqrt[3]{\sqrt{(x-y)^2}}=\sqrt{x+y}\cdot\sqrt[3]{|x-y|}=$$

If $x-y<0$, то $|x-y|=-(x-y)$ $=\sqrt{x+y}\cdot\sqrt[3]{-1\cdot(x-y)}=-\sqrt{x+y}\cdot\sqrt[3]{x-y};$

If $x-y>0$, то $\sqrt{x+y}\cdot\sqrt[3]{|x-y|}=\sqrt{x+y}\cdot\sqrt[3]{x-y}.$

The original system of equations is transformed into a set of systems:

$$\left[\begin{array}{l}\begin{cases}\sqrt{x+y}+\sqrt[3]{x-y}=6,\\ -\sqrt{x+y}\cdot\sqrt[3]{x-y}=8;\end{cases}\\ \begin{cases}\sqrt{x+y}+\sqrt[3]{x-y}=6,\\ \sqrt{x+y}\cdot\sqrt[3]{x-y}=8;\end{cases}\end{array}\right.$$

Нехай $\sqrt{x+y}=t, t\ge 0$, а $\sqrt[3]{x-y}=l$, тоді

$$\begin{cases}t+l=6,\\ -t\cdot l=8\end{cases}\to\begin{cases}l=6-t,\\ -t\cdot(6-t)=8\end{cases}\to\begin{cases}l=6-t,\\ -6t+t^2-8=0\end{cases}$$

$t^2-6t+8=0$ By Vieta's theorem, we have: $t_1=2, t_2=4.$

$l_1=6-2=4;\ l_2=6-4=2.$

$$\left[\begin{array}{l}\begin{cases}\sqrt{x+y}=2,\\ \sqrt[3]{x-y}=4\end{cases}\\ \begin{cases}\sqrt{x+y}=4,\\ \sqrt[3]{x-y}=2\end{cases}\end{array}\right. \quad \left[\begin{array}{l}\begin{cases}x+y=4,\\ x-y=64\end{cases} \begin{array}{l}2x=68,\\ x_1=34.\end{array}\\ \begin{cases}x+y=16,\\ x-y=8\end{cases} \begin{array}{l}2x=24,\\ x_2=12\end{array}\end{array}\right. \quad \left[\begin{array}{l}\begin{cases}y_1=4-34,\\ x_1=34\end{cases}\\ \begin{cases}y_1=12-8,\\ x_1=12\end{cases}\end{array}\right.$$

$$\left[\begin{array}{l}\begin{cases}y_1=-30,\\ x_1=34\end{cases}\\ \begin{cases}y_1=4,\\ x_1=12\end{cases}\end{array}\right.$$

– не виконується умова $x+y>0.$

We solve the second set of system:

$$\begin{cases}t+l=6,\\ t\cdot l=8\end{cases}\begin{cases}l=6-t,\\ t\cdot(6-t)=8\end{cases}\begin{cases}l=6-t,\\ -t^2+6t-8=0\end{cases}\begin{cases}l=6-t,\\ t^2-6t+8=0.\end{cases}$$

$D=36+32=68=4\cdot 17\quad t_1=\dfrac{6-2\sqrt{17}}{2}=3-\sqrt{17};\ t_2=3+\sqrt{17};$

$l_1=3+\sqrt{17};\ l_2=3-\sqrt{17}.$

$$\left[\begin{array}{l}\begin{cases}\sqrt{x+y}=3-\sqrt{17},\\ \sqrt[3]{x-y}=3+\sqrt{17},\end{cases} \begin{cases}x+y=9-6\sqrt{17}+17,\\ x-y=27+3\cdot 9\sqrt{17}+3\cdot 3\cdot 7+\left(\sqrt{17}\right)^3\end{cases} \begin{cases}x+y=26-6\sqrt{17},\\ x-y=180+44\sqrt{17}\end{cases}\\ \begin{cases}\sqrt{x+y}=3-\sqrt{17},\\ \sqrt[3]{x-y}=3-\sqrt{17},\end{cases} \begin{cases}x+y=26+6\sqrt{17},\\ x-y=27-27\sqrt{17}+103+17\sqrt{17}\end{cases} \begin{cases}x=78-13\sqrt{17},\\ y=-12+19\sqrt{17}\end{cases}\end{array}\right.$$

$2x=206+38\sqrt{17}, y_1=-76+44\sqrt{17},$

$x=103+19\sqrt{17}, y_2=-12+19\sqrt{17}.$

Answer: $(12;\ 4)$, $\left(103+19\sqrt{17};\ -76+44\sqrt{17}\right)$, $\left(78-19\sqrt{17};\ -12+19\sqrt{17}\right)$

$$\begin{cases}\dfrac{x-1}{2}=\dfrac{y+3}{3}=\dfrac{z-1}{4},\\ 2x+3y-5z+19=0.\end{cases}$$

Solution:

We denote $t=\dfrac{x-1}{2}=\dfrac{y+3}{3}=\dfrac{z-1}{4}$. From here: $2t=x-1,\ x=2t+1.$

$3t=y+3,\ y=3t-3,\ 4t=z-1,\ z=4t+1.$

$2\cdot(2t+1)+3\cdot(3t-3)-5\cdot(4t+1)+19=0,$

$4t+2+9t-9-20t-5+19=0,\ -7t+7=0,\ t=1;$

$x=2\cdot 1+1=3;\ \ y=3\cdot 1-3=0;\ \ z=4\cdot 1+1=5.$

Answer: $(3;0;5)$.

$$\begin{cases}x+y+z=4,\\ 2xy-z^2=16.\end{cases}$$

Solution:

Let us square the first equation and subtract the second equation:

$$-\begin{cases}x^2+y^2+z^2+2xy+2xz+2yz=16\\ 2xy-z^2=16\end{cases}$$

$x^2+y^2+2z^2+2xz+2yz=0,\qquad \left(x^2+2xz+z^2\right)+\left(y^2+2yz+z^2\right)=0,$

$(x+z)^2+(y+z)^2=0.$

This equality is possible only when

$(x+z)^2=0$ i $(y+z)^2=0$

$x+z=0\qquad y+z=0$

$x=-z;\qquad y=-z.$

Substituting these values into the first equation of the original system, we obtain:

$-z-z+z=4,\ \ z=4.$

$x_1=-4;\ \ y=-4.$

Answer: $(-4;-4;4)$.

The method of solving such a system of equations deserves special attention:

$$\begin{cases}(x)^{2y^2-1}=5,\\ x^{y^2+2}=125.\end{cases}$$

Solution:

Let us logarithm each of the equations of the system with base 5:

$$\begin{cases}(2y^2-1)\cdot\log_5 x=\log_5 x,\\(y^2+2)\cdot\log_5 x=\log_5 125.\end{cases}\quad \begin{cases}(2y^2-1)\cdot\log_5 x=1, & (1)\\(y^2+2)\cdot\log_5 x=3. & (2)\end{cases}\quad x>0.$$

Divide equation (2) by (1):

$$\frac{(y^2+2)\cdot\log_5 x}{(2y^2-1)\cdot\log_5 x}=\frac{3}{1};\ \frac{y^2+2}{2y^2-1}=\frac{3}{1};\ y^2+2=(2y^2-1)\cdot 3,\ y^2+2=6y^2-3,\ -5y^2=-5,$$

$y^2=1,\ y_1=-1;\ y_2=1.$

$x^{2\cdot(-1)^2-1}=5,\ x_1=5,\ x^{2\cdot 1^2-1}=5,\ x_2=5.$

Answer: $(5;-1),(5;1)$.

$$\begin{cases}2^x\cdot 3^y=6,\\3^x\cdot 4^y=12.\end{cases}$$

Solution:

$\begin{cases}2^x\cdot 3^y=2\cdot 3,\\3^x\cdot 4^y=2^2\cdot 3.\end{cases}$ Let us logarithm both equations with base 10:

$$\begin{cases}\lg(2^x\cdot 3^y)\lg(2\cdot 3),\\\lg(3^x\cdot 2^{2y})\lg(2^2\cdot 3).\end{cases}\quad\begin{cases}x\lg 2+y\lg 3=\lg 2+\lg 3,\\x\lg 3+2y\lg 2=2\lg 2+\lg 3.\end{cases}$$

We solve this system by the substitution method:
from the first equation, we express y in terms of x and substitute it into the second equation of the system:

$$y\lg 3=\lg 2+\lg 3-x\lg 2,\ y=\frac{\lg 2+\lg 3-x\lg 2}{\lg 3}.$$

Then the second equation will have the form:

$$x\lg 3+2\cdot\frac{\lg 2+\lg 3-x\lg 2}{\lg 3}\cdot\lg 2=2\lg 2+\lg 3\,\Big|\cdot\lg 3,$$

$$x\lg^2 3+2\lg^2 2+2\lg 2\cdot\lg 3-2\lg^2 2\cdot x=2\lg 2\cdot\lg 3+\lg^2 3;$$

$$x\cdot(\lg^2 3-2\lg^2 2)=\lg^2 3-2\lg^2 2;\ x=\frac{\lg^2 3-2\lg^2 2}{\lg^2 3-2\lg^2 2};\ x=1.$$

Value $x=1$ we substitute in the first equation of the original system of equations:

$$2^1\cdot 3^y=6,\ 3^y=\frac{6}{2},\ 3^y=3,\ y=1.$$

Answer: $(1;1)$.

$$\begin{cases}2-\log_2 y=2\log_2(x+y),\\\log_2(x+y)+\log_2(x^2-xy+y^2)=1.\end{cases}$$

Solution:

$2=\log_2 4;\ 1=\log_2 2.$

Range of valid values: $\begin{cases}y>0,\\x+y>0,\\x^2-xy+y^2>0.\end{cases}$

The system takes the form:

$$\begin{cases}\log_2 4-\log_2 y=\log_2(x+y)^2,\\ \log_2(x+y)+\log_2\left(x^2-xy+y^2\right)=\log_2 2.\end{cases}$$

We will potentiate both equations of the system:

$$\begin{cases}\dfrac{4}{y}=(x+y)^2,\\ (x+y)\cdot\left(x^2-xy+y^2\right)=2\end{cases}\quad \begin{cases}\dfrac{4}{y}=(x+y)^2, \quad y\neq 0.\\ x^3+y^3=2\end{cases}$$

We introduce a variable $y=tx.$ The system takes the form:

$$\begin{cases}4=tx\cdot(x+tx)^2,\\ 2=x^3+(tx)^3.\end{cases}\quad \begin{cases}4=tx\cdot\left(x^2+2x^2t+x^2t^2\right),\\ 2=x^3+x^3t^3.\end{cases}\quad \frac{4}{2}=\frac{tx^3\cdot\left(1+2t+t^2\right)}{x^3\cdot\left(1+t^3\right)};\ 2=\frac{t\cdot(1+t)^2}{(1+t)\left(-t+t^2\right)};$$

$$2=\frac{t+t^2}{1-t+t^2};\ t+t^2=2-2t+2t^2;\ 2t^2-3t^2-2t-t+2=0;\ t^2-3t+2=0;$$

$t_1=1;\ t_2=2.$

$y=1\cdot x,\ y=x;\ x^3+x^3=2;\ 2x^3=2;\ x^3=1;\ x=1.$

Then $y=1.$ *(1; 1)* - solution of the system of equations.

$$y=2\cdot x,\ x^3+(2x)^3=2;\ x^3+8x^3=2;\ 9x^3=2;\ x^3=\frac{9}{2};\ x=\sqrt[3]{\frac{2}{9}}=\sqrt[3]{\frac{2\cdot 3}{9\cdot 3}}=\frac{\sqrt[3]{6}}{3};\ y_2=\frac{2\sqrt[3]{6}}{3}.$$

Answer: $(1;1),\left(\dfrac{\sqrt[3]{6}}{3};\dfrac{2\sqrt[3]{6}}{3}\right).$

$$\begin{cases}y^{5x^2-51x+10}=1,\\ xy=15.\end{cases}$$

Solution:

$$1=y^0,\quad \begin{cases}y^{5x^2-51x+10}=y^0,\\ xy=15.\end{cases}\quad \begin{cases}5x^2-51x+10=0,\\ xy=15.\end{cases}$$

$$D=2601-200=2401=49^2,\ x_1=\frac{51-40}{10}=0{,}2;\ x_2=\frac{51+49}{10}=10.$$

$$y_1=\frac{15}{0{,}2}=75;\ y_2=\frac{15}{10}=1{,}5.$$

Answer: $(0{,}2;75),(10;1{,}5).$

$$\begin{cases}x^y=2, & (1)\\ (2x)^{y^2}=64. & (2)\end{cases}$$

Solution:

$x>0$ as the basis of the exponential function.

We transform the equation (2):

$(2x)^{y^2}=64,\ (2x)^{y^2}=2^6,\ 2^{y^2}\cdot x^{y^2}=2^6,\ 2^{y^2}\cdot\left(x^y\right)^y=2^6.$ Taking into account equality (1), we have:

$2^{y^2}\cdot 2^y=2^6,\ 2^{y^2+y}=2^6,\ y^2+y=6,\ y^2+y-6=0.$

By Vieta's theorem: $y_1=-3,\ y_2=2.$

$$x^{-3}=2,\ \frac{1}{x^3}=2;\ x^3=\frac{1}{2};\ x_1=\sqrt[3]{\frac{1}{2}}=\sqrt[3]{\frac{4}{8}}=\frac{\sqrt[3]{4}}{2};\ x^2=2,$$

$x=-\sqrt{2}$ – It does not satisfy the condition $x>0.$

$x_2=\sqrt{2}.$

Answer: $\left(\frac{\sqrt[3]{4}}{2};-3\right),\left(\sqrt{2};2\right).$

$$\begin{cases} y\cdot x^{\log_y x}=x^{2,5} & (1)\\ \log_3 y\cdot\log_y(y-2x)=1 & (2)\end{cases}$$

Solution:

Range of valid values: $\begin{cases} x>0,\\ y>0,\\ y>2x.\end{cases}$

Let us logarithm equation (1) with base 3:

$\log_3\left(y\cdot x^{\log_y x}\right)=\log_3 x^{2,5};$

$\log_3 y+\log_y x\cdot\log_3 x=2,5\cdot\log_3 x|\log_3 y,$

$(\log_3 y)^2+\log_y x\cdot\log_3 x\cdot\log_3 y=2,5\cdot\log_3 x\cdot\log_3 y;$

$(\log_3 y)^2+\frac{\log_3 x}{\log_3 y}\cdot\log_3 x\cdot\log_3 y-2,5\cdot\log_3 x\cdot\log_3 y=0;$

$(\log_3 y)^2-2,5\cdot\log_3 x\cdot\log_3 y+(\log_3 x)=0|:(\log_3 x\cdot\log_3 y);$

$\frac{\log_3 y}{\log_3 x}-2,5+\frac{\log_3 x}{\log_3 y}=0.$ We denote $\frac{\log_3 y}{\log_3 x}=t,$ then $t-25+\frac{1}{t}=0,$ $\begin{cases} t^2-2,5t+1=0,\\ t\neq 0\end{cases}$

$D=6,25-4=2,25.$

$t_1=\frac{2,5-1,5}{2}=\frac{1}{2};\ t_2=\frac{2,5+1,5}{2}=2.$

$$\left[\begin{array}{l}\frac{\log_3 y}{\log_3 x}=\frac{1}{2},\\ \frac{\log_3 y}{\log_3 x}=2.\end{array}\right.\ \left[\begin{array}{l}\log_3 x=2\log_3 y,\\ \log_3 y=2\log_3 x.\end{array}\right.\ \left[\begin{array}{l}\log_3 x=\log_3 y^2,\\ \log_3 y=\log_3 x^2.\end{array}\right.\ \left[\begin{array}{l}x=y^2,\\ y=x^2.\end{array}\right.$$

In equation (2) of the original system, we pass to the base of logarithms 3:

$\log_3 y\cdot\frac{\log_3(y-2x)}{\log_3 y}=1,\ \log_3(y-2x)=1,\ y-2x=3.$

If $x=y^2,$ то $y-2\cdot y^2-3=0,\ 2y^2-y+3=0$

$D=1-24=-23<0,\ \ y\in\varnothing.$

If $y=x^2,$ то $x^2-2x-3=0,\ x_1=-1$ does not satisfy the condition $x>0.$ $x_2-3.$

$y=2x+3=2\cdot 3+3=9.$

Answer: $(3;9).$

$$\begin{cases}\lg\sqrt{(x+y)^2}=1,\\ \lg y-\lg|x|=\lg 2.\end{cases}$$

Solution:

Range of valid values:

$\begin{cases} x+y\neq 0, \\ y>0, \\ x\neq 0. \end{cases}$ $\begin{cases} \lg|x+y|=1, \\ \lg y=\lg 2+\lg|x|. \end{cases}$ $x+y>0.$

$\begin{cases} |x+y|=10, \\ \lg y=\lg(2|x|). \end{cases}$ $\begin{cases} x+y=\pm 10, \\ y=2|x|. \end{cases}$ $\left[\begin{array}{l} \begin{cases} x+y=10, \\ y=2x. \end{cases} \\ \begin{cases} x+y=10, \\ y=-2x. \end{cases} \end{array}\right.$ $\begin{cases} x+2x=10, \\ y=2x. \end{cases}$ $\begin{cases} 3x=10, \\ y=2x. \end{cases}$ $\begin{cases} x=\dfrac{10}{3}, \\ y=\dfrac{20}{3}. \end{cases}$

$\begin{cases} x-2x=10, \\ y=-2x. \end{cases}$ $\begin{cases} -x=10, \\ y=20. \end{cases}$ $\begin{cases} x=-10, \\ y=20. \end{cases}$

Answer: $\left(\frac{10}{3};\frac{20}{3}\right), (-10;20).$

The solution of systems of trigonometric equations is reduced to one of three cases:

а) by means of identical transformations the system is reduced to one equation with one variable;

б) come to the system of equations with only arguments;

в) a system of equations is formed for the trigonometric functions of these arguments.

Such a system of trigonometric equations is very common:

$$\begin{cases} \sin x\cdot\cos x=-0{,}5, & (1) \\ \cos x\cdot\sin y=0{,}5. & (2) \end{cases}$$

Solution:

It is easy to see that the left-hand sides of equations (1) and (2) are parts of the formulas for the sine of the sum or difference of arguments. Hence follows a method for solving this system. Adding equations (1) and (2) and we obtain a new system of trigonometric equations:

$$\begin{cases} \sin x\cdot\cos y+\cos y\cdot\sin x=0, \\ \cos x\cdot\sin y-\sin x\cdot\cos y=1. \end{cases} \quad \begin{cases} \sin(x+y)=0, \\ \sin(y-x)=1. \end{cases}$$

Using separate cases of solving equations of the form $\sin x=a,$ we have:

This system is convenient to solve the method of adding:

$$\begin{cases} x+y=\pi n,\ n\in Z, \\ y-x=\dfrac{\pi}{2}+2k\pi,\ k\in Z. \end{cases}$$

$$2y=\frac{\pi}{2}+\pi n+2k\pi,\quad y=\frac{\pi}{4}+\frac{\pi n}{2}+k\pi.$$

Substitute the value y into the first equation of this system:

$$x+\frac{\pi}{4}+\frac{\pi n}{2}+k\pi=\pi n,\quad x=-\frac{\pi}{4}+\frac{\pi n}{2}-k\pi.$$

Answer: $\left(-\frac{\pi}{4}+\frac{\pi n}{2}-k\pi;\ \frac{\pi}{4}+\frac{\pi n}{2}+k\pi\right),\ n\in Z,\ k\in Z.$

$$\begin{cases} x - y = \dfrac{5\pi}{3}, \\ \sin x = 2\sin y. \end{cases}$$

Solution:

It is advisable to solve this system by the substitution method:

$$\begin{cases} y = x - \dfrac{5\pi}{3}, \\ \sin x = 2\cdot\left(\sin x\cdot\cos\dfrac{5\pi}{3} - \cos x\cdot\sin\dfrac{5\pi}{3}\right); \end{cases} \quad \begin{cases} y = x - \dfrac{5\pi}{3}, \\ \sin x = 2\cdot\left(\sin x\cdot\dfrac{1}{2} - \cos x\dfrac{\sqrt{3}}{2}\right); \end{cases}$$

$$\begin{cases} y = x - \dfrac{5\pi}{3}, \\ \sin x = \sin x - \sqrt{3}\cdot\cos x; \end{cases} \quad \begin{cases} y = x - \dfrac{5\pi}{3}, \\ \cos x = 0; \end{cases} \quad \begin{cases} y = x - \dfrac{5\pi}{3}, \\ x = \dfrac{\pi}{2} + \pi n,\ n \in Z. \end{cases}$$

$$y = \frac{\pi}{2} + \pi n - \frac{5\pi}{3} = \pi n + \frac{3\pi - 10\pi}{6} = \pi n - \frac{7\pi}{6},\ n \in Z.$$

Answer: $\left(\dfrac{\pi}{2} + \pi n;\ \pi n - \dfrac{7\pi}{6}\right),\ n \in Z.$

$$\begin{cases} tgx + tgy = 1, \\ x + y = \dfrac{\pi}{4}. \end{cases}$$

Solution:

In systems of equations of this type, it is necessary to look for the domain of definition of the system.

Because the $tg\alpha = \dfrac{\sin\alpha}{\cos\alpha}$, then the domain of the system:

$$\begin{cases} \cos x \neq 0, \\ \sin x \neq 0. \end{cases} \quad \begin{cases} x \neq \dfrac{\pi}{2} + k\pi,\ k \in Z, \\ y \neq \dfrac{\pi}{2} + \pi n,\ n \in Z. \end{cases}$$

The original system of equations is similar to the previous one, and therefore can be solved by the substitution method.

$$\begin{cases} y = \dfrac{\pi}{4} - x, \\ tgx + tg\left(\dfrac{\pi}{4} - x\right) = 1. \end{cases} \quad \begin{cases} y = \dfrac{\pi}{4} - x, \\ tgx + \dfrac{tg\dfrac{\pi}{4} - tgx}{1 + tg\dfrac{\pi}{4}\cdot tgx} = 1. \end{cases} \quad \begin{cases} y = \dfrac{\pi}{4} - x, \\ tgx + \dfrac{1 - tgx}{1 + tgx} = 1 \Big| \cdot (1 + tgx). \end{cases}$$

$$\begin{cases} y = \dfrac{\pi}{4} - x, \\ tgx + tg^2 x + 1 - tgx - tgx - 1 = 0. \end{cases} \quad \begin{cases} y = \dfrac{\pi}{4} - x, \\ tg^2 x - tgx = 0. \end{cases} \quad \begin{cases} y = \dfrac{\pi}{4} - x, \\ tg(tgx - 1) = 0. \end{cases}$$

This system is equivalent to such a set of systems:

$$\left[\begin{array}{l}\begin{cases} tgx = 0, \\ y = \dfrac{\pi}{4} - x; \end{cases} \\ \begin{cases} tgx - 1 = 0, \\ y = \dfrac{\pi}{4} - x; \end{cases}\end{array}\right. \quad \left[\begin{array}{l}\begin{cases} x = \pi n, \ n \in Z, \\ y = \dfrac{\pi}{4} - \pi n, \ n \in Z; \end{cases} \\ \begin{cases} x = \dfrac{\pi}{4} + k\pi, \ k \in Z, \\ y = \dfrac{\pi}{4} - \dfrac{\pi}{4} - k\pi, \ n \in Z. \end{cases}\end{array}\right.$$

Answer: $\left(\pi n; \ \dfrac{\pi}{4} - \pi n\right), \ n \in Z, \ \left(\dfrac{\pi}{4} + k\pi; \ -k\pi\right), \ k \in Z.$

$$\begin{cases} \sin x \cdot \sin y = 0{,}75, \\ tgx \cdot tgx = 3. \end{cases}$$

Solution:

We establish the range of permissible values:

$$\begin{cases} \cos x \neq 0, \\ \cos y \neq 0. \end{cases} \quad \begin{cases} x \neq \dfrac{\pi}{2} + \pi n, \ n \in Z, \\ x \neq \dfrac{\pi}{2} + k\pi, \ k \in Z. \end{cases}$$

We divide the first equation to the second:

$$\frac{\sin x \cdot \sin y}{tgx \cdot tgy} = \frac{0{,}75}{3}; \quad \frac{\sin x \cdot \sin y}{\dfrac{\sin x}{\cos x} \cdot \dfrac{\sin x}{\cos x}} = 0{,}25; \quad \cos x \cdot \cos y = 0{,}25 \ \text{(A)}$$

We replace the second equation of the system with the equation (A):

$$\begin{cases} \sin x \cdot \sin y = 0{,}75, \quad (B) \\ \cos x \cdot \cos y = 0{,}25. \quad (C) \end{cases}$$

Add the equations (B) and (C):

$\sin x \cdot \sin y + \cos x \cdot \cos y = 1$ *(D).*

Subtract the equation C and B:

$\cos x \cdot \cos y - \sin x \cdot \sin y = -0{,}5$ *(E)*

The left Part D and E equations represent the cosine of the difference and sum of the cosine respectively. We have a system of equations equivalent to the original system:

$$\begin{cases} \cos(x - y) = 1, \\ \cos(x + y) - 0{,}5. \end{cases} \quad \begin{cases} x - y = 2\pi n, \ n \in Z. \\ x + y = \pm \arccos(-0{,}5) + 2k\pi, \ k \in Z. \end{cases}$$

This system is equivalent to a set of systems:

$$\left[\begin{array}{l} + \begin{cases} x - y = 2\pi n, \\ x + y = \dfrac{2\pi}{3} + 2k\pi. \end{cases} \\ + \begin{cases} x - y = 2\pi n, \\ x + y = \dfrac{2\pi}{3} + 2k\pi. \end{cases} \end{array}\right. \quad \begin{array}{l} 2x = \dfrac{2\pi}{3} + 2k\pi + 2\pi n, \\ 2x = -\dfrac{2\pi}{3} + 2k\pi + 2\pi k. \end{array} \quad \left[\begin{array}{l} \begin{cases} x = \dfrac{\pi}{3} + k\pi + \pi n, \\ y = \dfrac{\pi}{3} + k\pi - 2\pi n. \end{cases} \\ \begin{cases} x = -\dfrac{\pi}{3} + k\pi + \pi k, \\ y = -\dfrac{\pi}{3} + k\pi - \pi n. \end{cases} \end{array}\right.$$

$$\begin{cases} x_1 = \dfrac{\pi}{3} + \pi(n+k), k \in Z. \\ y_1 = \dfrac{\pi}{3} + \pi(k-n), n \in Z. \end{cases} \quad \begin{cases} x_2 = -\dfrac{\pi}{3} + \pi(n+k), k \in Z. \\ y_2 = -\dfrac{\pi}{3} + \pi(k-n), n \in Z. \end{cases}$$

Answer: $\left(\dfrac{\pi}{3} + \pi(n+k); \dfrac{\pi}{3} + \pi(k-n)\right), n \in Z, k \in Z.$ $\left(-\dfrac{\pi}{3} + \pi(k+\pi); -\dfrac{\pi}{3} + \pi(k-n)\right).$

$$\begin{cases} \sin^2 x + \cos^2 x = 0{,}5, \\ x + y = \dfrac{\pi}{4}. \end{cases}$$

Solution:

Let us reduce the degree of sine and cosine of the first equation of the system:

$$\begin{cases} \dfrac{1-\cos 2x}{2} + \dfrac{1+\cos 2y}{2} = 0{,}5 \,\Big|\cdot 2; \\ x + y = \dfrac{\pi}{2}; \end{cases} \quad \begin{cases} 1 - \cos 2x + 1 + \cos 2y = 1, \\ x + y = \dfrac{\pi}{4}. \end{cases}$$

$$\begin{cases} \cos 2y - \cos 2x = -1, \\ x + y = \dfrac{\pi}{4}. \end{cases} \quad \begin{cases} -2\sin\dfrac{2y+2x}{2} \cdot \sin\dfrac{2y-2x}{2} = -1, \\ x + y = \dfrac{\pi}{4}. \end{cases}$$

$$\begin{cases} 2\sin(x+y) \cdot \sin(y-x) = 1, \\ x + y = \dfrac{\pi}{4}. \end{cases} \quad \begin{cases} 2\sin\dfrac{\pi}{4} \cdot \sin(x-y) = -1, \\ x + y = \dfrac{\pi}{4}. \end{cases}$$

$$2 \cdot \frac{\sqrt{2}}{2} \cdot \sin(x-y) = -1, \;\; \sin(x-y) = -\frac{1}{\sqrt{2}}, \;\; x - y = (-1)^n \cdot \left(-\frac{\pi}{4}\right) + 2k\pi.$$

$$+\begin{cases} x - y = (-1)^{n+1} \cdot \dfrac{\pi}{4} + 2\pi n, \\ x + y = \dfrac{\pi}{4}. \end{cases} \quad 2x = \frac{\pi}{4} + (-1)^{n+1} \cdot \frac{\pi}{4} + \pi n, \;\; x = \frac{\pi}{8} + (-1)^{n+1} \cdot \frac{\pi}{8} + \pi n, \;\; n \in Z.$$

$$y = \frac{\pi}{4} - \frac{\pi}{8} - (-1)^{n+1} \cdot \frac{\pi}{8} - \pi n = \frac{\pi}{8} \cdot \left(1 - (-1)^{n+1}\right) - \pi n, \;\; n \in Z.$$

Answer: $\left(\dfrac{\pi}{8} + (-1)^{n+1} \cdot \dfrac{\pi}{8} + \pi n\right); \left(\dfrac{\pi}{8} \cdot \left((-1)^{n+1}\right) - \pi n\right), n \in Z.$

Self-study assignments:

$$\begin{cases} x^2 y + xy^2 - x^2 - 4xy - 2y^2 + 3x + 4y = 2, \\ x - 2y = 1. \end{cases}$$

Answer: $(1; 0), (3; 1), \left(2; \dfrac{1}{2}\right).$

$$\begin{cases} x + y = 7, \\ xy = -18. \end{cases}$$

Answer: $(-2;9),(9;-2)$.

$$\begin{cases} x^2+y^2=20, \\ x+y=10. \end{cases}$$

Answer: $(6;4)$.

$$\begin{cases} x^2+y^2=20, \\ xy=8. \end{cases}$$

Answer: $(-2;-4),(2;4),(-4;-2),(4;2)$.

$$\begin{cases} x^2+2xy-8y^2-6x+18y-7=0, \\ 2x^2-5xy-10y^2-3x+9y+7=0. \end{cases}$$

Answer: $(-3;-1),(-1;2),(1;1),(3;1)$.

$$\begin{cases} x^3+y^3=1, \\ x^2y+2xy^2+y^2=2. \end{cases}$$

Answer: $\left(\frac{\sqrt[3]{3}}{3};\frac{2\sqrt[3]{3}}{3}\right),\left(\frac{\sqrt[3]{4}}{2};\frac{\sqrt[3]{4}}{2}\right)$.

$$\begin{cases} x^2+2xy-8y^2-6x+18y-7=0, \\ 2x^2-5xy-10y^2-3x+9y+7=0. \end{cases}$$

Answer: $(-3;-1),(-1;2),(1;1),(3;1)$.

$$\begin{cases} x^4+x^2y^2+y^4=481, \\ x^2+xy+y^2=37. \end{cases}$$

Answer: $(-3;-4),(-4;-3),(3;-4),(4;3)$.

$$\begin{cases} \sqrt[3]{x}+\sqrt[3]{y}=\frac{5}{2}\sqrt[6]{xy}, \ (xy\geq 0). \\ x+y=65. \end{cases}$$

Answer: $(1;64),(64;1)$.

$$\begin{cases} |x-y|+y^2=3, \\ |x-y|+|y-1|=2. \end{cases}$$

Answer: $(-1;1),(1;-1)$.

$$\begin{cases} \cos^2 x+\cos^2 y=1{,}5, \\ x+y=\pi. \end{cases}$$

Answer: $\pm\frac{5\pi}{6}+2\pi n; \quad \pm\frac{5\pi}{6}+\pi(1-2n), \ n\in Z.$

$$\begin{cases} x^2+2xy-7y^2 \ge \dfrac{1-a}{a+1}, \\ 3x^2+10xy-5y^2 \le -2. \end{cases}$$

Answer: $(-\infty;\ -1)$.

$$\begin{cases} x^{x-2y}=36, \\ 4(x-2y)+\log_6 x=9. \end{cases}$$ Find whole solutions.

Answer: $(6;\ 2)$.

$$\begin{cases} x+y+z=14, \\ x+yz=19. \end{cases}$$

Answer: $(5;2;7),(7;3;4),(7;4;3),(5;7;2)$.

For which a does the system have solutions $x>0,\ \ y>0$?

$$\begin{cases} ax+4y=6-9a, \\ 2x+(2+a)y=8. \end{cases}$$

Answer: $\left(-4;\dfrac{6}{13}\right)$.

$$\begin{cases} 2\log_x 2+8^{2\log_4 \sqrt[3]{3y}}=\log_{\sqrt{x}}\left(2x^2\right)-xy, \\ 2^x\cdot 4^y=8. \end{cases}$$

Answer: $\varnothing$.

$$\begin{cases} x^2+y^2=5, \\ xy=2. \end{cases}$$

Answer: $(-2;0)\cup(0;2)$.

Find positive solutions:

$$\begin{cases} x^{y+4x}=y, \\ x^3=y^{-1}. \end{cases}$$

Answer: $(-2;3)$.

$$\begin{cases} (2x-3y)^2+5\cdot(2x-3y)-6=0, \\ 2(x+y)^2-5(x+y)+2=0. \end{cases}$$

Answer: $\left(\dfrac{7}{5};\dfrac{3}{5}\right)$, $\left(\dfrac{1}{2};0\right)$, $(0;2)$, $\left(-\dfrac{9}{10};\dfrac{7}{5}\right)$.

$$\begin{cases} x^2+y^2=34, \\ x+y+xy=23. \end{cases}$$

Answer: $(3;5),(5;3)$.

$$\begin{cases} 2^x \cdot 3^y = 12, \\ 2^y \cdot 3^x = 18. \end{cases}$$

Answer: $(2;1)$.

$$\begin{cases} xy + x + y = 11, \\ x^2 y + xy^2 = 30. \end{cases}$$

Answer: $(3;2),(5;1),(1;5),(2;3)$.

$$\begin{cases} 2x + y = 7, \\ xy = 6. \end{cases}$$

Answer: $\left(\frac{3}{2};4\right),(4;-1)$.

$$\begin{cases} x - y = 1, \\ x^3 - y^3 = 7. \end{cases}$$

Answer: $(-1;-2),(2;-1)$.

$$\begin{cases} x^2 + y^2 = 5, \\ xy = 2. \end{cases}$$

Answer: $(1;2),(2;1),(-1;-2),(-2;-1)$.

$$\begin{cases} x^2 - 2xy - 3y^2 = 0, \\ x^2 - xy - 2x - 3y = 6. \end{cases}$$

Answer: $(-2;2),\left(\frac{3}{2};-\frac{3}{2}\right),(6;2),\left(-\frac{3}{2};-\frac{1}{2}\right)$.

$$\begin{cases} \log_5(x + y) = 1, \\ 2^x + 2^y = 12. \end{cases}$$

Answer: $(2;3),(3;2)$.

$$\begin{cases} \dfrac{1}{x-1} + \dfrac{2}{y+1} = 1\dfrac{1}{6}, \\ \dfrac{3}{x-1} - \dfrac{1}{y+1} = 1\dfrac{1}{6}. \end{cases}$$

Answer: $(3;2)$.

$$\begin{cases} x^2 - 4xy + y^2 = 3, \\ y^2 - 3xy = 2. \end{cases}$$

Answer: $\left(-\frac{\sqrt{2}}{2};\frac{\sqrt{2}}{2}\right),\left(\frac{\sqrt{2}}{2};-\frac{\sqrt{2}}{2}\right).$

$$\begin{cases} x^2-6xy+8y^2=0, \\ x^2+3y^2-xy=45. \end{cases}$$

Answer: $(6;\ 3),\ (-6;\ -3),\ (4\sqrt{3};\ \sqrt{3}),\ (-4\sqrt{3};\ -\sqrt{3}).$

$$\begin{cases} \log_y x+\log_y x=2, \\ x^2-y=20. \end{cases}$$

Answer: $(5;\ 5).$

$$\begin{cases} x+y+\sqrt{x+y}=20, \\ x^2+y^2=136. \end{cases}$$

Answer: $(10;\ 6),\ (6;\ 10).$

$$\begin{cases} \sqrt[3]{x}+\sqrt[3]{y}=4, \\ x+y=28. \end{cases}$$

Answer: $(27;\ 1),\ (1;\ 27).$

$$\begin{cases} \log_3 x+\log_3 y=2+\log_3 2, \\ \log_{27}(x+y)=\frac{2}{3}. \end{cases}$$

Answer: $(6;\ 3),\ (3;\ 6).$

Write in response $7x-2y$:

$$\begin{cases} 5^x\cdot 2^y=3200, \\ \log_{\sqrt{5}}(y-x)=2. \end{cases}$$

In response, write down the product of roots:

$$\begin{cases} x^{\lg y}=4, \\ xy=40. \end{cases}$$

Write in response $15y+2x$:

$$\begin{cases} \frac{1}{x}-\frac{1}{y}=\frac{2}{15}, \\ \log_3 x+\log_3 y=1+\log_3 5. \end{cases}$$

Write in response $2x-y$ for $0<x\le 90^\circ$:

$$\begin{cases} \sin^2 x + \sin^2 y = \dfrac{7}{4}, \\ x + y = \dfrac{5\pi}{6}. \end{cases}$$

For which *a* does the system have no solutions?

$$\begin{cases} 2x + 31y = -23, \\ 2x + 2ay = 23. \end{cases}$$

Answer: 15,5.

In response, write down the product of roots:

$$\begin{cases} 2^x \cdot 3^y = 6, \\ 3^x \cdot 4^y = 12. \end{cases}$$

Record a piece in response $x \cdot y$:

$$\begin{cases} \sqrt[3]{x + y} = 2, \\ (x + y) \cdot 7^x = 2744. \end{cases}$$

Write the largest value in response *y*:

$$\begin{cases} (1 + y)^x = 100, \\ \left(y^4 - 2y^2 + 1\right)^{x-1} = \dfrac{(y-1)^{2x}}{(y+1)^2}. \end{cases}$$

In response, write down the largest sum of roots:

$$\begin{cases} x + y + xy = 11, \\ x + y - xy = 1. \end{cases}$$

In response, write down the product of roots:

$$\begin{cases} \sqrt{\dfrac{20y}{x}} = \sqrt{x + y} + \sqrt{x - y}, \\ \sqrt{\dfrac{16x}{5y}} = \sqrt{x + y} - \sqrt{x - y}. \end{cases}$$

In response, write *xy* for $x > 0,\ y > 0$:

$$\begin{cases} y^{x+y} = x^4, \\ x^{x+y} = y. \end{cases}$$

Answer: 1.

For which *m* the system has no solutions:

$$\begin{cases} 3x+(m-2)y=1, \\ (m+2)x+4y=2. \end{cases}$$

Answer: -4.

Write in response $6x-7y$:

$$\begin{cases} \dfrac{1}{2}\log_2 x-\log_4 y=0, \\ x^2-5y^2+4=0. \end{cases}$$

Answer: -1.

$$\begin{cases} 3^{-x}\cdot 2^y=1152, \\ \log_{\sqrt{5}}(x+y)=2. \end{cases}$$

Answer: $(-2;\ 7)$.

$$\begin{cases} 3^x\cdot 2^y=972, \\ \log_{\sqrt{5}}(x-y)=2. \end{cases}$$

Answer: $(5;\ 2)$.

$$\begin{cases} \log_9\left(x^3+y^3\right)=\log_3(x+y), \\ \log_3\left(x^2-y^2\right)=\log_3(x+y). \end{cases}$$

Answer: $(1;\ 0)$, $(2;\ 1)$.

$$\begin{cases} (x+y)\cdot 3^{y-x}=\dfrac{5}{27}, \\ 3\log_5(x+y)=x-y. \end{cases}$$

Answer: $(4;\ 1)$.

$$\begin{cases} \log_2\dfrac{x^2\sqrt{y+1}}{2}=2, \\ \log_8 x\cdot\log_2(y+1)^2=\dfrac{4}{3}. \end{cases}$$

Answer: $(2;\ 3)$, $\left(\sqrt{2};\ 15\right)$.

$$\begin{cases} y^2+2x=7, \\ x-3y=0. \end{cases}$$

Answer: $(-21;\ -7)$, $(3;\ 1)$.

$$\begin{cases} x^2-y=1, \\ x-y=-5. \end{cases}$$

Answer: $(3;\ 8)$, $(-2;\ 3)$.

$$\begin{cases} x^3 + y^3 = 9, \\ xy = 2. \end{cases}$$

Answer: $(1;\ 2),\ (2;\ 1)$.

Find product of roots:

$$\begin{cases} 3 \cdot \left(2\log_{y^2} x + \log_{\frac{1}{x}} y \right) = 10, \\ xy = 81. \end{cases}$$

Write in response $\dfrac{x}{y}$:

$$\begin{cases} \log_2 x = \dfrac{1}{2}\log_2 y + \dfrac{1}{2}\log_2 (4 - x), \\ \log_2 (x + y) = \dfrac{\log_3 \left(\dfrac{y}{x} \right)}{\log_3 \dfrac{1}{3}}. \end{cases}$$

$$\begin{cases} tgx - tgy = -2\sqrt{3}, \\ x - y = \dfrac{\pi}{3}. \end{cases}$$

Answer: $\left(\dfrac{2\pi}{3} + k\pi;\ \dfrac{\pi}{3} + k\pi \right),\ k \in Z.$

$$\begin{cases} \sin\dfrac{x+y}{2} \cdot \sin\dfrac{x-y}{2} = \dfrac{\sqrt{2}}{2}, \\ \cos x \cdot \cos y = -\dfrac{1}{2}. \end{cases}$$

Answer: $\left(\pm\dfrac{3\pi}{4} + 2\pi n;\ \ \pm\dfrac{\pi}{4} + 2\pi n \right),\ n \in Z.$

$$\begin{cases} x - y = -\dfrac{\pi}{6}, \\ \sin x \cdot \sin y = \dfrac{\sqrt{3}}{4}. \end{cases}$$

Answer: $\left(\dfrac{\pi}{6} + \dfrac{k\pi}{2};\ \dfrac{\pi}{3} + \dfrac{k\pi}{2} \right),\ k \in Z.$

$$\begin{cases} x + y = \dfrac{\pi}{3}, \\ tgx \cdot tgy = \dfrac{1}{3}. \end{cases}$$

Answer: $\left(\frac{\pi}{6}+k\pi;\ \frac{\pi}{6}-k\pi\right),\ k\in Z.$

$$\begin{cases} \sin x\cdot\sin y=\frac{1}{4}, \\ 3tgx=tgy. \end{cases}$$

Answer: $\left(\frac{\pi}{6}+(k+n)\pi;\ \frac{\pi}{3}+(k-n)\pi\right),\ k\in Z,\ n\in Z.$

$$\begin{cases} x\cdot\sin^2\left(x-\frac{\pi}{6}\right)=y\cdot\cos^2\left(y+\frac{\pi}{6}\right), \\ x\cdot\cos^2\left(x-\frac{\pi}{6}\right)=y\cdot\sin^2\left(y+\frac{\pi}{6}\right). \end{cases}$$

Answer: $(0;\ 0),\ \left(\frac{\pi}{4}+\frac{k\pi}{2};\ \frac{\pi}{4}+\frac{k\pi}{2}\right),\ k\in Z.$

The solution of systems of equations with three or more variables will require, in addition to classical methods, some artificial methods. This is due to the fact that it is not always possible to express one variable well in terms of others and thus reduce the number of unknowns in the system.
And the second reason - is not always transformed system of equations with fewer variables is solved relatively easily.
Let's illustrate this with specific exercises..

$$\begin{cases} x\cdot(y-1)=3, & (1) \\ (3-y)\cdot z=1, & (2) \\ (x-2)\cdot(2-z)=1. & (3) \end{cases}$$

Solution:

From equations (1) and (2), we express the variables x and z through y:

$x=\frac{3}{y-1};\ z=\frac{1}{3-y}$ (4); The values found substitute in equation (3):

$\left(\frac{3}{y-1}-2\right)\cdot\left(2-\frac{1}{3-y}\right)=1.$ We solve the resulting equation:

$y\neq 1$ i $y\neq 3.$ $\frac{3-2y+2}{y-1}\cdot\frac{6-2y-1}{3-y}=1;\ \frac{5-2y}{y-1}\cdot\frac{5-2y}{3-y}=1;$

$(5-2y)^2=(y-1)\cdot(3-y);\ 25-20y+4y^2=y^2+3y+y-3;\ 5y^2-24y+28=0;$

$D=24^2-4\cdot 5\cdot 28=576-560=16.$

$y_1=\frac{24-4}{10}=2;\ y_2=\frac{24+4}{10}=2{,}8.$

Substituting these values into formulas (4), we find: x and z values:

$x_1=\frac{3}{2-1}=3;\ x_2=\frac{3}{2-2{,}8}=\frac{3}{-0{,}8}=-\frac{3}{\frac{4}{5}}=-\frac{15}{4};$

$$z_1=\frac{1}{3-2}=1;\ z_2=\frac{1}{3-2{,}8}=\frac{1}{0{,}8}=5.$$

Answer: $(3;\ 2;\ 1)$, $\left(-\frac{15}{4};\ 2{,}8;\ 5\right)$.

$$\begin{cases} x-y+z=0, & (1)\\ 3x^2+3z^2+5xyz=0, & (2)\\ 2x^3-2y^3-3xyz=0. & (3)\end{cases}$$

Solution:

From the first equation, we express z through x and y:

$z=y-x.$

Substituting this value of z into the second and third equations of the original system, we obtain a system of two equations in two variables:

$$\begin{cases} 3x^2+3\cdot(y-x)^2+5xy\cdot(y-x)=0, & (4)\\ 2x^3-2y^3-3xy\cdot(y-x)=0. & (5)\end{cases}$$

The second equation of this system can be transformed:

$2\cdot(x-y)\cdot(x^2+xy+y^2)+3xy\cdot(x-y)=0;$

$(x-y)\cdot(2x^2+2xy+2y^2+3xy)=0;$

$(x-y)\cdot(2x^2+5xy+2y^2)=0;$

$$\left[\begin{aligned} &x-y=0,\\ &2x^2+5xy+2y^2=0.\end{aligned}\right.$$

$2x^2+5xy+2y^2=0 \,|: xy;\ 2\left(\frac{x}{y}\right)+5+2\left(\frac{y}{x}\right)=0.$

We denote $\frac{x}{y}=t$:

$2\cdot t+5+\frac{2}{t}=0,\ \ t\neq 0.$

$2t^2+5t+2=0,\ \ D=25-16=9.$

$t_1=\frac{-5-3}{4}=-2;\ t_2=\frac{-5+3}{4}=-\frac{1}{2}.$

$x=-2y,\ \ x=-0{,}5y.$

We form a set of three systems of equations:

$$\left[\begin{aligned} &\begin{cases} 3x^2+3(y-x)+5xy(y-x)=0,\\ x-y=0.\end{cases} \quad (6)\\ &\begin{cases} 3x^2+3(y-x)+5xy(y-x)=0,\\ x=-2y.\end{cases} \quad (7)\\ &\begin{cases} 3x^2+3(y-x)+5xy(y-x)=0,\\ x=-0{,}5y.\end{cases} \quad (8)\end{aligned}\right.$$

Let's solve the system of equations (6):

$3x^2+3\cdot 0+5xy\cdot 0=0,\ x_1=0;\ y_1=0.$

Let's solve the system of equations (7):

$3\cdot(-2y)^2+3\cdot(y+2y)+5\cdot(-2y)\cdot y\cdot(y+2y)=0,$

$12y^2+9y-30y^3=0|:(-3)$

$10y^3-4y^2-3y=0$

$y=0$ або $y^2-4y-3=0,\ D=16+12=28=4\cdot 7$

$y_2=\dfrac{4-2\sqrt{7}}{2}=2-\sqrt{7};\ y_3=2+\sqrt{7};\ x_2=-4+2\sqrt{7};\ x_3=-4-2\sqrt{7}.$

System (8) has solutions:

$3\cdot(-0{,}5y)^2+3\cdot(y+0{,}5y)+5\cdot(-0{,}5y)\cdot y\cdot(y+0{,}5y)=0;$

$0{,}75y^2+4{,}5y-3{,}75y^3=0;|:(-0{,}75);\ 5y^3-y^2-6y=0;$

$y\cdot(5y^2-y-6)=0;\ y=0$ або $5y^2-y-6=0;$

$D=1+120=121;\ y_4=\dfrac{1-11}{10}=-1;\ y_5=\dfrac{1+11}{10}=1{,}2;$

$x_4=-0{,}5\cdot(-1)=0{,}5;\ x_5=-0{,}5\cdot 1{,}2=-0{,}6.$

$z_1=0-0=0;\ z_2=2-\sqrt{7}-(-4+2\sqrt{7})=2-\sqrt{7}+4-2\sqrt{7}=6-3\sqrt{7};$

$z_3=2+\sqrt{7}-(-4-2\sqrt{7})=2+\sqrt{7}+4+2\sqrt{7}=6+3\sqrt{7};$

$z_4=-1-0{,}5=-1{,}5;\ z_5=1{,}2+0{,}6=1{,}8.$

Answer: $(0;\ 0;\ 0)$, $(-4+2\sqrt{7};2-\sqrt{7};6-3\sqrt{7})$, $(-4-2\sqrt{7};\ 2+\sqrt{7};\ 6+3\sqrt{7})$, $(0{,}5;\ -1;\ -1{,}5)$, $(-0{,}6;\ 1{,}2;\ 1{,}8)$.

$$\begin{cases} y+z=3, & (1)\\ z+x=-5, & (2)\\ x+y=4. & (3)\end{cases}$$

Solution:

Let's add these equations:

$2x+2y+2z=2\,|:2;\ x+y+z=1\ \ (4).$

Substituting into equation (4) the value $z+y$, we get $3+x=1,\ x=-2.$

Similarly $y-5=1,\ y=6;\ z+4=1,\ z=-3.$

Answer: $(-2;\ 6;\ -3)$.

$$\begin{cases} yz=-4, & (1)\\ zx=3, & (2)\\ xy=27. & (3)\end{cases}$$

Solution:

We multiply the left sides of equations (1 - 3), as well as their right sides:

$x^2y^2z^2=-324;\ xyz=\sqrt{-324}.$

Answer: $\varnothing$.

$$\begin{cases} zx+xy=-5, & (1)\\ xy+yz=4, & (2)\\ yz+xz=3. & (3)\end{cases}$$

Solution:

Add equations (1 – 3):

$2xy+2xz+2yz=2|:2,\ \ xy+xz+yz=1\ \ (4)$

Substituting successively equations (1 - 3) into (4), we obtain a new system of equations:

$$\begin{array}{l} zy-5=1, \\ xz+4=1, \\ xy+3=1. \end{array} \quad \begin{cases} zy=6 & (5) \\ xz=-3 & (6) \\ xy=-2 & (7) \end{cases}$$

We solve this system of equations similarly:

$(xyz)^2=36, \ \ xyz=\pm 6$ (8)

Substituting successively (6 - 7) into (8), we obtain:

$$\begin{array}{l} x\cdot 6=\pm 6; \\ y\cdot(-3)=\pm 6; \\ x\cdot(-2)=\pm 6. \end{array} \quad \begin{cases} x=\pm 1; \\ y=\pm 2; \\ z=\pm 3. \end{cases}$$

Answer: $(1;\ 2;\ 3),\ (1;\ -2;\ 3),\ (1;\ -2;\ -3),\ (-1;\ 2;\ 3),\ (-1;\ -2;\ 3),\ (-1;\ -2;\ -3).$

$$\begin{cases} (y+z)\cdot(z+x)=15, & (1) \\ (z+x)\cdot(x+y)=10, & (2) \\ (x+y)\cdot(y+z)=6. & (3) \end{cases}$$

Solution:

Let us multiply the left and right equations of the system:

$(x+y)^2\cdot(x+z)^2\cdot(y+z)^2=900; \ \ ((x+y)\cdot(x+z)\cdot(y+z))^2=900.$

$(x+y)\cdot(x+z)\cdot(y+z)=\pm 30;$ (4)

Substituting successively (1 - 3) into (4), we obtain a set of two systems of equations:

$$\left[\begin{array}{l} \begin{cases} x+y=2, \\ y+z=3, \\ z+x=5. \end{cases} \\ \begin{cases} x+y=-2, \\ y+z=-3, \\ z+x=-5. \end{cases} \end{array}\right. \quad \left[\begin{array}{l} 2x+2y+2z=10. \\ \\ 2x+2y+2z=-10. \end{array}\right. \quad \left[\begin{array}{l} x+y+z=5. \\ \\ x+y+z=-5. \end{array}\right. \quad \begin{cases} 2+z=5, \\ x+3=5. \end{cases} \quad \begin{cases} y+5=5, \\ -2+z=-5, \\ x-3=-5, \\ y-5=-5. \end{cases} \quad \begin{cases} z=3, \\ x=2, \\ y=0. \end{cases} \quad \begin{cases} y=0, \\ z=-3, \\ x=-2. \end{cases}$$

Answer: $(2;\ 0;\ 3),\ (-2;\ 0;\ -3).$

$$\begin{cases} z+x-y=3, & (1) \\ x+y-z=-5, & (2) \\ y+z-x=1. & (3) \end{cases}$$

Solution:

Add equations (1) and (2):

$$+\begin{array}{l} z+x-y=3, \\ x+y-z=-5. \\ \hline 2x=-2;\ \ x=-1. \end{array}$$

Add equations (2) and (3):

$$+\begin{cases} x+y-z=-5, \\ y+z-x=1. \end{cases}$$
$$\overline{\ \ 2y=-4;\ \ y=-2.}$$

$$+\begin{cases} z+x-y=3, \\ y+z-x=1. \end{cases}$$

$$2z=4;\ z=2.$$

Answer: $(-1;\ -2;\ 2)$.

$$\begin{cases} x^2-(y-z)^2=3, \\ y^2-(z-x)^2=12, \\ z^2-(x-y)^2=1. \end{cases}$$

Solution:

To each of the equations of the system, we apply the formula for the difference of squares:

$$\begin{cases} (x-y+z)\cdot(x+y-z)=3, & (1) \\ (y-z+x)\cdot(y+z-x)=12, & (2) \\ (z-x+y)\cdot(z+x-y)=4. & (3) \end{cases}$$

We multiply the left and right sides of the equations of the system:

$$(x+y-z)^2\cdot(y+z-x)^2\cdot(z+x-y)^2=144,$$

$$(x+y-z)\cdot(y+z-x)\cdot(z+x-y)=\pm 12\ \ (4)$$

Substituting (1 - 3) in (4) in succession, we obtain two systems of equations:

$$\begin{cases} (y+z-x)\cdot 3=12, \\ (z+x-y)\cdot 12=12, \\ (x+y-z)\cdot 4=12. \end{cases} \text{ i } \begin{cases} (y+z-x)\cdot 3=-12, \\ (z+x-y)\cdot 12=-12, \\ (x+y-z)\cdot 4=-12. \end{cases}$$

$$+\begin{cases} y+z-x=4, & (5) \\ z+x-y=1, & (6) \\ x+y-z=3. & (7) \end{cases}$$

(5)+(6):

$2z=5,\ z=2{,}5.$

(6)+(7):

$2x=4,\ x=2.$

(5)+(7):

$2y=7,\ y=3{,}5.$

$$+\begin{cases} y+z-x=-4, & (8) \\ z+x-y=-1, & (9) \\ x+y-z=-3. & (10) \end{cases}$$

(8)+(9):

$2z=-5,\ z=-2{,}5.$

(9)+(10):

$2x=-4,\ x=-2.$

(8)+(10):

$2y=-7,\ y=-3{,}5.$

Answer: $(2;\ 3{,}5;\ 2{,}5)$, $(-2;\ -3{,}5;\ -2{,}5)$.

$$\begin{cases} \dfrac{xyz}{x+y}=1, \\ \dfrac{2xyz}{y+z}=1, \\ \dfrac{5xyz}{z+x}=1. \end{cases}$$

Solution:

Overturn each of the equation of system:

$$\begin{cases} \dfrac{x+y}{xyz}=1, \\ \dfrac{y+z}{2xyz}=1 \Big| \cdot 2, \\ \dfrac{z+x}{5xyz}=1 \Big| \cdot 5. \end{cases} \quad \begin{cases} \dfrac{x+y}{xyz}=1, \\ \dfrac{y+z}{2xyz}=2, \\ \dfrac{z+x}{5xyz}=5. \end{cases}$$

We divide each term of the numerator of the fraction by the denominator, assuming that $x \neq 0, y \neq 0, z \neq 0.$

$$\begin{cases} \dfrac{1}{yz}+\dfrac{1}{xz}=1, & (1) \\ \dfrac{1}{xz}+\dfrac{1}{xy}=2, & (2) \\ \dfrac{1}{yz}+\dfrac{1}{xy}=5. & (3) \end{cases}$$

After adding the left and right sides of the equations of the system, we get:

$$\frac{2}{yz}+\frac{2}{xz}+\frac{2}{xy}=8 \Big| :2 \quad \frac{1}{xy}+\frac{1}{xz}+\frac{1}{yz}=4. \quad (4)$$

Substituting (1 - 3) into (4) we get:

$$\begin{cases} \dfrac{1}{xy}+1=4, \\ \dfrac{1}{yz}+2=4, \\ \dfrac{1}{xz}+5=4. \end{cases} \quad \text{Звідси:} \begin{cases} \dfrac{1}{xy}=3, \\ \dfrac{1}{yz}=2, \\ \dfrac{1}{xz}=-1. \end{cases} \quad \begin{cases} xy=\dfrac{1}{3}, & (5) \\ yz=\dfrac{1}{2}, & (6) \\ xz=-1. & (7) \end{cases}$$

After multiplying the left and right sides of the equation of the last system, we get:

$$(xyz)^2=-\frac{1}{6} \quad \varnothing.$$

Answer: $\varnothing$.

$$\begin{cases} x^2+y^2+z^2=29, & (1) \\ xy+yz+zx=26, & (2) \\ xy-yz-zx=-14. & (3) \end{cases}$$

Multiplying both sides of equation (2) by 2:

$$+\begin{array}{l} 2xy+2yz+2xz=52 \ (4). \\ x^2+\ y^2+\ \ z^2=29 \end{array}$$ Add equations (1) and (4):

$$\overline{x^2+y^2+z^2+2xy+2yz+2xz=81.}$$

By the formula for the square of the trinomial, we have:

$(x+y+z)^2=81, \ x+y+z=\pm 9.$

Add equations (2) and (3):

$2xy=12, \ xy=6.$

We form two systems of equations:

$$\begin{cases} x+y+z=9, \\ xy=6, \\ xy+yz+zx=26. \end{cases} \quad \text{and} \quad \begin{cases} x+y+z=-9, \\ xy=6, \\ xy+yz+xz=26. \end{cases}$$

$$\begin{cases} x+y=9-z, \\ 6+z\cdot(x+y)=26. \end{cases} \qquad \begin{cases} x+y=-9-z, \\ 6+z\cdot(y+x)=26. \end{cases}$$

$$6+z\cdot(9-z)=26, \qquad 6+z\cdot(-9-z)=26,$$

$$6-9z-z^2-26=0,$$

$$6+9z-z^2-26=0, \qquad z^2+9z+20=0,$$

$$z^2-9z+20=0, \qquad D=81-80=1,$$

$$z_1=4,\ z_2=5; \qquad z_1=\frac{-9-1}{2}=-5,\ z_2=\frac{-9+1}{2}=-4;$$

We have two systems of equations:

$$\begin{cases} x+y+4=9, \\ xy=6. \end{cases} \begin{cases} x+y+5=9, \\ xy=6. \end{cases} \qquad \begin{cases} x+y-5=-9, \\ xy=6. \end{cases} \begin{cases} x+y-4=-9, \\ xy=6. \end{cases}$$

$$\begin{cases} x+y=5, \\ xy=6. \end{cases} \quad \begin{cases} x+y=4, \\ xy=6. \end{cases} \qquad \begin{cases} x+y=-4, \\ xy=6. \end{cases} \quad \begin{cases} x+y=-5, \\ xy=6. \end{cases}$$

$$\begin{cases} y=5-x, \\ x\cdot(5-x)-6=0. \end{cases} \begin{cases} y=4-x, \\ x\cdot(4-x)-6=0. \end{cases} \qquad \begin{cases} y=-4-x, \\ x\cdot(-4-x)=6. \end{cases} \begin{cases} y=-5-x, \\ x\cdot(-5-x)=6. \end{cases}$$

$$-x^2+5x-6=0,\quad -x^2+4x-6=0,\quad -4x-x^2-6=0,\quad -5x-x^2-6=0$$

$$x^2-5x+6=0,\quad x^2-4x+6=0,\quad x^2+4x+6=0,\quad x^2+5x+6=0$$

$$x_1=2,\ x_2=3. \quad D=16-24<0 \quad D=16-24<0 \quad D=25-24=1$$

$$y_1=3,\ y_2=2. \quad \varnothing \quad \varnothing \quad x_3=\frac{-5-1}{2}=-3;\ x_4=\frac{-5+1}{2}=-2.$$

$$z_1=4,\ z_2=4. \qquad y_3=-5+3=2;\ y_4=-5+2=3.$$

$$z_3=-5,\ z_4=-5.$$

Answer: $(2;\ 3)$, $(3;\ 2)$, $(-3;\ -2)$, $(-2;\ -3)$.

$$\begin{cases} x+y-z=2, & (1) \\ x^2+y^2+z^2=6, & (2) \\ x^3+y^3-z^3=8. & (3) \end{cases}$$

Solution:

We transform the third equation of the system:

$x^3+y^3=8+z^3; \to (x+y)\cdot(x^2-xy+y^2)=8+z^3$ (4).

From the first equation of the system we have:

$$\begin{cases} x+y=2+z, & (5) \\ x^2+y^2=6-z^2, & (6) \\ (x+y)^2=(2+z)^2. & (8) \end{cases}$$

It follows from the second equation that $(x+y)^2=x^2+2xy+y^2$,

$2xy=(x+y)^2-(x^2+y^2)$;

$$xy=\frac{(x+y)^2-(x^2+y^2)}{2} \quad (7)$$

Substitute (6) and (8) in (7):

$$xy=\frac{(2+z)^2-(6-z^2)}{2}=\frac{4+4z+z^2-6+z^2}{2}=\frac{4z+2z-2}{2}=z^2+2z-1 \quad (9)$$

Substitute (5), (6), (9) in (4):

$$(2+z)\cdot(6-z^2-z^2-2z+1)=(2+z)\cdot(4-2z+z^2);$$
$$(2+z)\cdot(7-2z^2-2z)-(2+z)\cdot(4-2z+z^2)=0;$$
$$(2+z)\cdot(7-2z^2-2z-4+2z-z^2)=0;$$
$$(2+z)\cdot(-3z^2+3)=0;$$
$$2+z=0, \qquad -3z^2+3=0,$$
$$z_1=-2. \qquad z^2-1=0,\ (z-1)\cdot(z+1)=0,\ z_2=-1;\ z_3=1.$$

Formed a combination of three systems of equations:

$$\begin{cases}x+y=0,\\ x^2+y^2=2,\\ z=-2.\end{cases} \begin{cases}x+y=1,\\ x^2+y^2=5,\\ z=-1.\end{cases} \begin{cases}x+y=3,\\ x^2+y^2=5,\\ z=-1.\end{cases}$$

Let's solve each of these systems of equations:

$$y=-x,\quad x^2+(-x)^2=2,\ 2x^2=2,\ x^2=1,\ x_1=-1,\ x_2=1.$$
$$y_1=1,\ y_2=-1.$$
$$y=1-x,\ (1-x)^2+x^2=5,\ x^2+1-2x+x^2=5,\ 2x^2-2x-4=0 \,|:2\quad x^2-x-2=0,$$
$$x_3=-1,\ x_4=2.$$
$$y_3=2,\ y_4=-1.$$
$$y=3-x,\ (3-x)^2+x^2=5,\ 9-6x+x^2+x^2-5=0,\ 2x^2-6x+4=0 \,|:2\quad x^2-3x+2=0,$$
$$x_5=1,\ x_6=2.$$
$$y_5=2,\ y_6=1.$$

Answer: $(-1;\ 1), (1;\ -1), (-1;\ 2), (2;\ -1), (1;\ 2), (2;\ 1)$.

$$\begin{cases}x+y+z=-2, & (1)\\ \dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}=-\dfrac{1}{2}, & (2)\\ xyz=2. & (3)\end{cases}$$

Solution:

We transform equation (2) taking into account equation (3):

$$\frac{yz+xz+xy}{xyz}=-\frac{1}{2},\ \frac{yz+xz+xy}{2}=-\frac{1}{2},\ yz+xz+xy=-1.$$

We form a system of equations equivalent to the original:

$$\begin{cases}x+y+z=-2, & (4)\to\\ yz+xz+xy=-1, & (5)\to\\ xyz=2. & (6)\end{cases} \begin{cases}x+y=-z-2, & (7)\\ (x+y)\cdot z+xy=-1, & (8)\\ xy=\dfrac{2}{z}. & (9)\end{cases}$$

Substituting (7) and (9) in (8), we obtain:

$$(-z-2)\cdot z+\frac{2}{z}+1=0 \,|\cdot z,\ -z^3-2z^2+z+2=0\to z^3+2z^2-z-2=0.$$

We are looking for divisors of the free member: ± 1 and ± 2.

-1: $(-1)^3+2\cdot(-1)^2+1-2=-1+2+1-2=0$. $z_1=-1$.

$$\begin{array}{l|l} z^3+2z^2-z-2 & z+1 \\ \hline -\;z^3+z^2 & z^2+z-2 \\ \quad z^2-z & \\ \quad -\;z^2+z & \\ \qquad -2z-2 & \\ \qquad -2z-2 & \\ \qquad\quad 0 & \end{array}$$

$z^2+z-2=0$, $D=1+8=9$.

$z_2=\dfrac{-1-3}{2}=-2$; $z_3=\dfrac{-1+3}{2}=1$.

Substitute $z_1=-1$ into equation (7) and (9):

$\begin{cases} x+y=1-2, \\ xy=\dfrac{2}{-1}. \end{cases}$ $\begin{cases} x+y=-1, \\ xy=-2. \end{cases}$ $\begin{cases} y=-1-x, \\ x\cdot(-1-x)=-2. \end{cases}$ $\begin{cases} y=-1-x, \\ -x-x^2+2=0. \end{cases}$

$x^2+x-2=0$. Substitute $z=-2$ into equation (7) and (9):

$\begin{cases} x_1=-2, \\ y_1=1, \\ z_1=-1. \end{cases}$ $\begin{cases} x_2=1, \\ y_2=-2, \\ z_2=-1. \end{cases}$ $\begin{cases} x+y=2-2, \\ xy=\dfrac{-2}{2}. \end{cases}$ $\begin{cases} x+y=0, \\ xy=-1. \end{cases}$ $\begin{cases} y=-x, \\ x\cdot(-x)=-1. \end{cases}$ $\begin{cases} y=-x, \\ -x^2=-1. \end{cases}$

$x_3=1$, $y_3=-1$, $z_3=-2$, $x_4=-1$, $y_4=1$, $z_4=-2$.

Substitute $z=1$ into equation (7) and (9), we obtain:

$\begin{cases} x+y=-1-2, \\ xy=\dfrac{2}{1}. \end{cases}$ $\begin{cases} x+y=-3, \\ xy=2. \end{cases}$ $\begin{cases} y=-3-x, \\ x\cdot(-3-x)=2. \end{cases}$ $\begin{cases} y=-3-x, \\ -3x-x^2-2=0. \end{cases}$

$\begin{cases} y=-3-x, \\ x^2+3x+2=0. \end{cases}$ $x_5=-1$, $x_6=-2$, $y_5=-2$, $y_6=-1$, $z_5=1$, $z_6=1$.

Answer: $(-2;\ 1;\ -1)$, $(1;\ -2;\ -1)$, $(1;\ -1;\ -2)$, $(-1;\ 1;\ -2)$, $(-1;\ -2;\ 1)$, $(-2;\ -1;\ 1)$.

This system of equations could be solved using the generalized Vieta theorem, but it is not included in the mathematics curriculum for schools.

$$\begin{cases} x^2+xy+y^2=1, & (1) \\ y^2+yz+z^2=3, & (2) \\ z^2+zx+x^2=7. & (3) \end{cases}$$

Solution:

From equation (1) we subtract equation (2):

$x^2-y^2+y(x-z)+y^2-z^2=-2 \rightarrow x^2-z^2+y(x-z)=-2$,

$(x-z)\cdot(x+z)+y\cdot(x-z)=-2$, $(x-z)\cdot(x+y+z)=-2$ (4)

Similarly (2) - (3):

$y^2-z^2+z\cdot(y-x)+z^2-x^2=-4 \rightarrow y^2-x^2+z\cdot(y-x)=-4$.

$(y-x)\cdot(y+x)+z(y-x)=-4 \rightarrow (y-x)\cdot(x+y+z)=-4$ (5)

Divide (4) by (5):

$$\frac{(x-z)\cdot(x+y+z)}{(y-x)\cdot(x+y+z)}=\frac{-2}{-4}\to\frac{x-z}{y-x}=\frac{1}{2}\to 2x-2z=y-x,\quad y=3x-2x\quad (6)$$

Substitute (6) in (1):

$$x^2+x\cdot(3x-2z)+(3x-2z)^2=1\to x^2+3x^2-2xz+9x^2-12xz+4z^2=1,$$

$$\begin{cases}13x^2-14xz+4z^2=1,\\ x^2+xz+z^2=7\,|\cdot 14\end{cases}\quad +\begin{cases}13x^2-14xz+4z^2=1,\\ 14x^2+14xz+14z^2=98\end{cases}$$

$$27x^2+18z^2=99$$

Consider the system of equations:

Substitution $z=tx.$

$$\begin{cases}13x^2-14x(tx)+4(tx)^2=1,\\ x^2+x(tx)+(tx)^2=7.\end{cases}\quad\begin{cases}13x^2-14x^2t+4x^2t^2=1,\\ x^2+x^2t+x^2t^2=7.\end{cases}$$

$$\left\{\frac{x^2\cdot(13-14t+4t^2)=1,}{x^2\cdot(1+t+t^2)=7.}\right.\quad\frac{13-14t+4t^2}{1+t+t^2}=\frac{1}{7};$$

Using the main property of proportion, we get:

$$7\cdot(13-14t+t^2)=(1+t+t^2)\cdot 1\to 91-98t+28t^2=1+t+t^2;$$

$$91-98t+28t^2-1-t-t^2=0\to 27t^2-99t+90=0;$$

$$D=99^2-4\cdot 27\cdot 90=9801-9720=81\to t_1=\frac{99-9}{54}=\frac{90}{54}=\frac{5}{3};\ t_2=\frac{108}{54}=2.$$

$z=\frac{5}{3}x$ або $z=2x.$ Substitute these values into (6):

$$y=3x-2\cdot\frac{5}{3}x=3x-\frac{10}{3}x=\frac{9x-10x}{3}=-\frac{1}{3}x\ \text{or}\ y=3x-2\cdot 2=3x-4.$$

$$(A)\begin{cases}27x^2+11z^2=99,\\ y=-\frac{1}{3}x,\\ z=\frac{5}{3}x.\end{cases}\quad(Б)\begin{cases}27x^2+11z^2=99,\\ y=-\frac{1}{3}x,\\ z=2x.\end{cases}\quad(В)\begin{cases}27x^2+11z^2=99,\\ y=3x-4,\\ z=\frac{5}{3}x.\end{cases}\quad(Г)\begin{cases}27x^2+11z^2=99,\\ y=3x-4,\\ z=2x.\end{cases}$$

Let's solve each of these systems of equations:

A): $27x^2+11\cdot\left(\frac{5}{3}x\right)^2=99;\quad 27x^2+\frac{25}{9}x^2=99\,|\cdot 9;\quad 243x^2+25x^2=891,$

$$268x^2=891,\quad x^2=\frac{891}{263},\quad x_1=-\sqrt{\frac{891}{263}},\quad x_2=\sqrt{\frac{891}{263}}.$$

$$y_1=\frac{1}{3}\sqrt{\frac{891}{263}},\quad y_2=-\frac{1}{3}\sqrt{\frac{891}{263}},\quad z_1=-\frac{5}{3}\sqrt{\frac{891}{263}},\quad z_2=\frac{5}{3}\sqrt{\frac{891}{263}}.$$

B): $27x^2+11\cdot(2x)^2=99;\quad 27x^2+44x^2=99;\quad 71x^2=99,$

$$x_3=-\sqrt{\frac{99}{71}},\quad x_4=\sqrt{\frac{99}{71}}.$$

$$y_3=\frac{1}{3}\sqrt{\frac{99}{71}},\quad y_4=-\frac{1}{3}\sqrt{\frac{99}{71}},\quad z_3=-2\sqrt{\frac{99}{71}},\quad z_4=2\sqrt{\frac{99}{71}}.$$

C): $x_5=-\sqrt{\frac{891}{263}},\ x_6=\sqrt{\frac{891}{263}},$

$y_5=-3\sqrt{\frac{891}{263}}-4,\ y_6=3\sqrt{\frac{891}{263}}-4,\ z_5=-\frac{5}{3}\sqrt{\frac{891}{263}},\ z_6=\frac{5}{3}\sqrt{\frac{891}{263}}.$

D): $x_7=-\sqrt{\frac{99}{71}},\ x_8=\sqrt{\frac{99}{71}},$

$y_7=-3\sqrt{\frac{99}{71}}-4,\ y_8=3\sqrt{\frac{99}{71}}-4,\ z_7=-2\sqrt{\frac{99}{71}},\ z_8=2\sqrt{\frac{99}{71}}.$

Answer:

$\left(-\sqrt{\frac{891}{263}};\frac{1}{3}\sqrt{\frac{891}{263}};-\frac{5}{3}\sqrt{\frac{891}{263}}\right),\left(\sqrt{\frac{891}{263}};-\frac{1}{3}\sqrt{\frac{891}{263}};\frac{5}{3}\sqrt{\frac{891}{263}}\right),\left(-\sqrt{\frac{99}{71}};\frac{1}{3}\sqrt{\frac{99}{71}};-2\sqrt{\frac{99}{71}}\right),$

$\left(\sqrt{\frac{99}{71}};-\frac{1}{3}\sqrt{\frac{99}{71}};2\sqrt{\frac{99}{71}}\right),\left(-\sqrt{\frac{891}{263}};-3\sqrt{\frac{891}{263}}-4;-\frac{5}{3}\sqrt{\frac{891}{263}}\right),\left(\sqrt{\frac{891}{263}};3\sqrt{\frac{891}{263}}-4;\frac{5}{3}\sqrt{\frac{891}{263}}\right),$

$\left(-\sqrt{\frac{99}{71}};-3\sqrt{\frac{99}{71}}-4;-2\sqrt{\frac{99}{71}}\right),\left(\sqrt{\frac{99}{71}};3\sqrt{\frac{99}{71}}-4;2\sqrt{\frac{99}{71}}\right).$

$$\begin{cases}(x+y)\cdot(x+z)=x, & (1)\\ (y+z)\cdot(y+x)=2y, & (2)\\ (z+x)\cdot(z+y)=3z. & (3)\end{cases}$$

Solution:

A unique way to solve.

If $x=0,$ то $(y+z)\cdot(y+x)=2y,\ (y+z)\cdot y=2y,\ y^2+yz=2y,\ y+z=2.$

Equation (1) implies $yz=0\Rightarrow y\neq 0, z=0; (0;0;0).$

Let the $x=y=0,$ then from equation (3) we have:

$(z+0)\cdot(z+0)=3z,\ z^2=3z,\ z\cdot(z-3)=0,\ z=0$ або $z=3;\ (0;0;3).$

Let the $x=z=y,$ then $(y+0)\cdot(y+0)=2y,\ y^2-2y=0,\ y^2-2y=0,\ y=0$ or $y=2;$ $(0;2;0).$

Let the $y=z=0,$ then $(x+0)\cdot(x+0)=x,\ x^2-x=0,\ x=0,\ x=1;\ (1;0;0).$

Let the $x\neq 0, y\neq 0, z\neq 0.$

Divide (1) by (2):

$$\frac{(x+y)\cdot(x+z)=x}{(y+z)\cdot(x+y)=2y};\ \frac{x+z}{y+z}=\frac{x}{2y}.$$

Divide (2) by (3):

$$\frac{(y+z)\cdot(y+x)=2y}{(z+x)\cdot(z+y)=3z};\ \frac{y+x}{z+x}=\frac{2y}{3z}.$$

Applying the main property of proportion, we get:

$$\begin{cases}(x+z)\cdot 2y=(y+z)\cdot x, \rightarrow\\ (y+x)\cdot 3z=(z+x)\cdot 2y.\rightarrow\end{cases}\begin{cases}2xy+2zy=yx+xz,\\ 3yz+3xz=2yz+2xy.\end{cases}$$

$$\begin{cases}xy+2yz-xz=0, & (4)\\ 2xy-yz-3xz=0. & (5)\end{cases}\rightarrow+\begin{cases}xy+2yz-xz=0,\\ 4xy-2yz-6xz=0.\end{cases}$$

$5xy-7xz=0,\ x(5y-7y)=0, x\neq 0,$ тоді $5y-7z=0,\ y=\frac{7}{5}z.$

We substitute this value into the equation (4):

$x\cdot\frac{7}{5}z+2\cdot\frac{7}{5}z\cdot z-xz=0,\ \frac{7}{5}zx+\frac{14}{5}z^2-xz=0,\ \frac{2}{5}zx+\frac{14}{5}z^2=0;\ z\cdot\left(\frac{2}{5}x+\frac{14}{5}z\right)=0, z\neq 0,$

then $\frac{2}{5}x+\frac{14}{5}z=0|\times 5,\ 2x=-14z,\ x=-7z.$

We substitute the values of *x* and *y* into the equation (3):

$(z-7z)\cdot\left(z+\frac{7}{5}z\right)=3z,\ -6z\cdot 2{,}4z=3z|:3z, -2z\cdot 2{,}4=1;$

$z=-1:4\frac{8}{10}=1:\left(-4\frac{4}{5}\right)=-1:\frac{24}{5}=-\frac{5}{24};$

$y=\frac{7}{5}\cdot\left(-\frac{5}{24}\right)=-\frac{7}{24};\ x=-7\cdot\left(-\frac{5}{24}\right)=-\frac{35}{24}.$

Answer: $(0;0;0),(0;0;3),(0;2;0),(1;0;0),\left(\frac{35}{24};-\frac{7}{24};-\frac{5}{24}\right).$

$$\begin{cases}3x-\sqrt[4]{x+7y}=2y,\\ \sqrt{x+y}+2y=3x.\end{cases}$$

Solution:

$\begin{cases}\sqrt[4]{x+7y}=3x-2y,\\ \sqrt{x+y}=3x-2y.\end{cases}$ It is advisable to replace $3x-2y=a$ (1)

Then $\begin{cases}\sqrt[4]{x+7y}=a,\\ \sqrt{x+y}=a.\end{cases}\ \begin{cases}x+7y=a^4,\\ x+y=a^2|\cdot(-1).\end{cases}\ \begin{cases}x+7y=a^4,\\ -x-y=-a^2|\cdot 7.\end{cases}$

Adding the equation of the last system, we get the value of *x* and *y*:

$6y=a^4-a^2,\ y=\frac{a^4-a^2}{6},$

$\begin{cases}x+7y=a^4,\\ x+y=a^2|\cdot(-7)\end{cases}\ +\begin{cases}x+7y=a^4,\\ -7x-7y=7a^2\end{cases}$ $\quad x=\frac{7a^2-a^4}{6}.$

$-6x=a^4-7a^2$

Substituting these values of *x* and *y* in the replacement (1), we obtain:

$3\cdot\frac{7a^2-a^4}{6}-2\cdot\frac{a^4-a^2}{6}=a|\cdot 6,\ 21a^2-3a^4-2a^4+2a^2=6a,$

$-5a^4+23a^2-6a=0|\cdot(-1),\ 5a^4-23a^2+6a=0,\ a\cdot(5a^3-23a+6)=0,\ a_1=0;$

$5a^3-23a+6=0.\ a_2=2,$ бо $5\cdot 2^3-23\cdot 2+6=0.$

$$\begin{array}{r|l} 5a^3-23a+6 & a-2 \\ \hline -\;5a^3-10a^2 & 5a^2+10a-3 \\ \hline 10a^2-23a & \\ -\;10a^2-20a & \\ \hline 3a+6 & \\ -\;3a+6 & \\ \hline 0 & \end{array}$$

$D=100+60=160,$

$$a_1=\frac{-10-\sqrt{160}}{10}=\frac{-10-2\sqrt{40}}{10}=$$

$$=\frac{-10-4\sqrt{10}}{10}=-\frac{5+2\sqrt{10}}{5}<0,\ a>0.$$

$$a_3=\frac{-5+2\sqrt{10}}{5}>0.$$

Then $x_1=\dfrac{7\cdot 0^2-2^4}{6}=0;\ y_1=\dfrac{0^4-0^2}{6}=0.$ $\quad(0;\ 0).$

$x_2=\dfrac{7\cdot 2^2-2^4}{6}=2;\ y_2=\dfrac{2^4-2^2}{6}=\dfrac{16-4}{6}=2.$ $\quad(2;\ 2).$

$$x_3=\frac{7\cdot\left(\frac{-5+2\sqrt{10}}{5}\right)^2-\left(\frac{-5+2\sqrt{10}}{5}\right)^4}{6}=\frac{7\cdot\frac{25-20\sqrt{10}+40}{25}-\left(\frac{65-20\sqrt{10}}{25}\right)^2}{6}=$$

$$=\frac{\frac{455\cdot 140\sqrt{10}}{25}-\frac{4225-2600\sqrt{10}+4000}{625}}{6}=\frac{\frac{11375-3500\sqrt{10}-4225+2600\sqrt{10}-4000}{625}}{6}=$$

$$=\frac{\frac{3150-900\sqrt{10}}{625}}{6}=\frac{3150-900\sqrt{10}}{3750}=\frac{315-90\sqrt{10}}{375}=\frac{63-18\sqrt{10}}{75}=\frac{21-6\sqrt{10}}{25}.$$

$$y_3=\frac{a^4-a^2}{6}=\frac{\left(\frac{2\sqrt{10}-5}{5}\right)^4-\left(\frac{2\sqrt{10}-5}{5}\right)^2}{6}=\frac{\left(\frac{40-20\sqrt{10}+25}{25}\right)^2-\frac{40-20\sqrt{10}+25}{25}}{6}=$$

$$=\frac{\left(\frac{65-20\sqrt{10}}{25}\right)^2-\left(\frac{65-20\sqrt{10}}{25}\right)}{6}=\frac{\frac{4225-2600\sqrt{10}+4000}{625}-\frac{56-20\sqrt{10}}{25}}{6}=$$

$$=\frac{8295-2600\sqrt{10}-1625+500\sqrt{10}}{625\cdot 6}=\frac{6600-2100\sqrt{10}}{3750}=\frac{660-210\sqrt{10}}{375}=$$

$$=\frac{132-42\sqrt{10}}{75}=\frac{44-14\sqrt{10}}{25}.$$

Answer: $(0;0),(2;2),\left(\dfrac{21-6\sqrt{10}}{25};\dfrac{44-14\sqrt{10}}{25}\right).$

$$\frac{y+z-x}{4}=\frac{x+z-y}{16}=\frac{x+y-z}{4}=xyz.$$

Solution:

Let's introduce a new variable: $\dfrac{y+z-x}{4}=\dfrac{x+z-y}{16}=\dfrac{x+y-z}{4}=xyz=t.$

We form the system of equations:

$$\begin{cases} y+z-x=4t, & (1) \\ x+z-y=16t, & (2) \\ x+y-z=4t, & (3) \\ xyz=t. & (4) \end{cases}$$

Add equations (1), (2), (3):

$x+y+z=24t$ (5)

Subtract (1) from (5): $2x=20t,\ x=10t.$

Subtract (2) from (5): $2y=8t,\ y=4t.$

Subtract (3) from (5): $2z=20t,\ z=10t.$

We substitute the found values of x, y and z into the equation (4):

$10t\cdot 4t\cdot 10t=t;\ \ 400t^3-t=0,\ \ t\cdot\left(400t^2-1\right)=0;$

$t_1=0,\ \ x_1=0,\ \ y_1=0,\ \ z_1=0.$

$t^2=\dfrac{1}{400};\ \ x_2=-\dfrac{1}{2},\ y_2=-\dfrac{1}{5},\ z_2=-\dfrac{1}{2}.$

$t_2=-\dfrac{1}{20};\ \ x_3=\dfrac{1}{2},\ y_3=\dfrac{1}{5},\ z_3=\dfrac{1}{2}.$

$t_3=\dfrac{1}{20}.$

Answer: $(0;0;0),\left(-\dfrac{1}{2};-\dfrac{1}{5};-\dfrac{1}{2}\right),\left(\dfrac{1}{2};\dfrac{1}{5};\dfrac{1}{2}\right).$

Self-study assignments:

$\begin{cases} x^2+xy=15, \\ y^2+xy=10. \end{cases}$ Answer: $(3;2),(-3;-2).$

$\begin{cases} y^2+xy=231, \\ x^2+xy=210. \end{cases}$ Answer: $(10;11),(-10;-11).$

$\begin{cases} x^3+y^3=35, \\ x+y=5. \end{cases}$ Answer: $(2;3),(3;2),(-2;-3),(-3;-2).$

$\begin{cases} x\cdot y=12, \\ xz=15, \\ yz=20. \end{cases}$ Answer: $(3;4;5),(-3;-4;-5).$

$\begin{cases} xy+xz=7, \\ xy+yz=15, \\ yz+xz=16. \end{cases}$ Answer: $(1;3;4),(-1;-3;-4).$

$\begin{cases} x^2+xy-xz=2, \\ y^2+xy-yz=3, \\ z^2-xy-yz=4. \end{cases}$ Answer: $\left(\dfrac{2}{3};1;-\dfrac{4}{3}\right),\left(-\dfrac{2}{3};-1;\dfrac{4}{3}\right).$

$$\begin{cases} \dfrac{xy}{x+y}=\dfrac{3}{4}, \\ \dfrac{xz}{x+z}=\dfrac{5}{6}, \\ \dfrac{yz}{y+z}=\dfrac{15}{8}. \end{cases}$$ Answer: $(1;3;5)$.

$$\begin{cases} x+y+z=11, \\ xy+xz+yz=36, \\ xyz=36. \end{cases}$$ Answer: $(2;3;6),(2;6;3),(3;2;6),(3;6;2),(6;2;3),(6;3;2)$.

$$\begin{cases} x^2+y^2=z^2, \\ xy+xz+yz=47, \\ (z-x)\cdot(z-y)=2. \end{cases}$$

Answer: $(-4;-3;-5),(-3;-4;-5),(4;3;5),(3;4;5)$,

$\left(\dfrac{7-\sqrt{113}}{2};\dfrac{7+\sqrt{113}}{2};9\right),\left(\dfrac{7+\sqrt{113}}{2};\dfrac{7-\sqrt{113}}{2};9\right)$,

$\left(\dfrac{-7-\sqrt{113}}{2};\dfrac{-7+\sqrt{113}}{2};-9\right),\left(\dfrac{-7+\sqrt{113}}{2};\dfrac{-7-\sqrt{113}}{2};-9\right)$.

$$\begin{cases} x^2-6xy+6y^2=-2, \\ x^2-6xz+11z^2=3, \\ y^2-4yz+3z^2=0. \end{cases}$$

Answer: $(-6;-1;-1),(-2;-1;-1),(2;1;1),(6;1;1),(-4;-3;-1),(4;3;1)$,

$\left(-\dfrac{46}{\sqrt{337}};-\dfrac{15}{\sqrt{337}};-\dfrac{5}{\sqrt{337}}\right),\left(\dfrac{46}{\sqrt{337}};\dfrac{15}{\sqrt{337}};\dfrac{5}{\sqrt{337}}\right)$.

$$\begin{cases} \dfrac{x-1}{5}=\dfrac{y-2}{3}=\dfrac{z-2}{2}, \\ x+2y-z=12. \end{cases}$$ Answer: $(6;5;4)$.

$$\begin{cases} x+\sqrt{\dfrac{x}{y}}=\dfrac{2}{y}, \\ y-x=3. \end{cases}$$ Answer: $\left(\dfrac{-3+\sqrt{13}}{2};\dfrac{3+\sqrt{13}}{2}\right),(-4;-1)$.

$$\begin{cases} \sqrt[3]{x}+\sqrt[3]{y}=\dfrac{5}{2}\sqrt[6]{xy}, \\ x+y=65. \end{cases}$$ Answer: $(1;64),(64;1)$.

$$\begin{cases} \sqrt{\dfrac{x}{y}}+\sqrt{\dfrac{y}{x}}=\dfrac{7}{\sqrt{xy}}+1, \\ x\sqrt{xy}+y\sqrt{xy}=78. \end{cases}$$ Answer: $(4;9),(9;4)$.

$$\begin{cases} \sqrt{\dfrac{6x}{x+y}}+\sqrt{\dfrac{x+y}{6x}}=\dfrac{5}{2}, \\ xy=x+y. \end{cases}$$ Answer: $\left(3;\dfrac{3}{2}\right),\left(\dfrac{24}{23};24\right)$.

$$\begin{cases}\sqrt{x+\dfrac{1}{y}}+\sqrt{x+y-3}=-3,\\ 2x+y+\dfrac{1}{y}=8.\end{cases}$$ Answer:

$(3;1),(5;-1),\left(4+\sqrt{10};3-\sqrt{10}\right),\left(4-\sqrt{10};3+\sqrt{10}\right).$

$$\begin{cases}\dfrac{x+\sqrt{x^2-y^2}}{x-\sqrt{x^2-y^2}}+\dfrac{x-\sqrt{x^2-y^2}}{x+\sqrt{x^2-y^2}}=\dfrac{17}{4},\\ x\cdot(x+y)+\sqrt{x^2+xy+4}=52.\end{cases}$$ Answer: $(5;4),(-5;-4),(15;-12),(-15;12).$

$$\begin{cases}x^2+y\cdot\sqrt[3]{x^2y}=68,\\ y^2+x\cdot\sqrt[3]{xy^2}=17.\end{cases}$$ Answer: $(8;1),(-8;-1),(8;-1),(-8;1).$

$$\begin{cases}11y+\sqrt[5]{x+9y}=7x,\\ \sqrt[3]{x+y}=7x-11y.\end{cases}$$ Answer:

$(0;0),(5;3),(-5;-3),\left(\dfrac{10}{243};-\dfrac{1}{243}\right),\left(-\dfrac{10}{243};\dfrac{1}{243}\right).$

$$\begin{cases}\log_2(x+y)=2,\\ \log_{\sqrt{3}}y+\log_{\sqrt{3}}x=2.\end{cases}$$ Answer: $(3;1),(1;3).$

$$\begin{cases}2^x\cdot 3^y=24,\\ 2^y\cdot 3^x=54.\end{cases}$$ Answer: $(3;1).$