Section 23
Calculating boundaries

$\frac{1}{\infty}=0;\ \frac{1}{0}=\infty.$

Theorems on limits:

1. $\lim\limits_{n\to\infty}(x_n \pm y_n)=\lim\limits_{n\to\infty}x_n \pm \lim\limits_{n\to\infty}y_n;$
2. $\lim\limits_{n\to\infty}(x_n \cdot y_n)=\lim\limits_{n\to\infty}x_n \cdot \lim\limits_{n\to\infty}y_n;$

Consequence: $\lim\limits_{n\to\infty}(k \cdot x_n)=k\cdot\lim\limits_{n\to\infty}x_n;$

3. $\lim\limits_{n\to\infty}\left(\frac{x_n}{y_n}\right)=\frac{\lim\limits_{n\to\infty}x_n}{\lim\limits_{n\to\infty}y_n}.$

Theorems on boundary transition

1. $\lim\limits_{n\to\infty}(x_n^n)=\left(\lim\limits_{n\to\infty}x_n\right)^n.$
2. $\lim\limits_{n\to\infty}\sqrt[m]{x_n}=\sqrt[m]{\lim\limits_{n\to\infty}x_n}.$
3. При $a>0;\ x_n>0$ $\lim\limits_{n\to\infty}\log_a x_n=\log_a\left(\lim\limits_{n\to\infty}x_n\right).$
4. При $a>0$ $\lim\limits_{n\to\infty}a^{x_n}=a^{\lim\limits_{n\to\infty}x_n}.$
5. $\lim\limits_{n\to\infty}\frac{1}{n^2}=0;\ \lim\limits_{n\to\infty}\frac{1}{n}=0.$

Calculating the Limit of a Number Sequence

In the exercises below, you need to divide the numerator and denominator of the fraction by the variable to the highest degree that is included in this fraction.

$$\lim_{n\to\infty}\frac{1+3n+2n^2}{1-n^2}=\lim_{n\to\infty}\frac{\frac{1+3n+2n^2}{n^2}}{\frac{1-n^2}{n^2}}=\lim_{n\to\infty}\frac{\frac{1}{n^2}+\frac{3}{n}+2}{\frac{1}{n^2}-1}=$$

$$=\frac{\lim\limits_{n\to\infty}\left(\frac{1}{n^2}+\frac{3}{n}+2\right)}{\lim\limits_{n\to\infty}\left(\frac{1}{n^2-1}\right)}=\frac{\lim\limits_{n\to\infty}\frac{1}{n^2}+\lim\limits_{n\to\infty}\frac{3}{n}+\lim\limits_{n\to\infty}2}{\lim\limits_{n\to\infty}\frac{1}{n^2}-\lim\limits_{n\to\infty}1}=\frac{0+0+2}{0-1}=-2.$$

Solution Brief:

$$\lim_{n\to\infty}\frac{7n^2+2n-3}{5n^2-4n+4}=\lim_{n\to\infty}\frac{\frac{7n^2}{n^2}+\frac{2n}{n^2}-\frac{3}{n^2}}{\frac{5n^2}{n^2}-\frac{4n}{n^2}+\frac{4}{n^2}}=\lim_{n\to\infty}\frac{7+\frac{2}{n}-\frac{3}{n^2}}{5-\frac{4}{n}+\frac{4}{n^2}}=$$

$$=\frac{\lim\limits_{n\to\infty}7+\lim\limits_{n\to\infty}\frac{2}{n}-\lim\limits_{n\to\infty}\frac{3}{n^2}}{\lim\limits_{n\to\infty}5-\lim\limits_{n\to\infty}\frac{4}{n}+\lim\limits_{n\to\infty}\frac{4}{n^2}}=\frac{7+0-0}{5-0+0}=\frac{7}{5}=1{,}4.$$

$$\lim_{n\to\infty}\frac{(2n-1)\cdot(3n+2)\cdot(4n-3)}{5n^2+n+1}=\lim_{n\to\infty}\frac{24n^3-14n^2-11n+6}{5n^2+n+1}=$$

$$=\lim_{n\to\infty}\frac{\frac{24n^3}{n^3}-\frac{14n^2}{n^3}-\frac{11n}{n^3}+\frac{6}{n^3}}{\frac{5n^2}{n^3}+\frac{n}{n^3}+\frac{1}{n^3}}=\lim_{n\to\infty}\frac{24-\frac{14}{n}-\frac{11}{n^2}+\frac{6}{n^3}}{\frac{5}{n}+\frac{1}{n^2}+\frac{1}{n^3}}=\frac{24-0-0+0}{0+0+0}=\infty$$ (here $n\neq 0$)

$$\lim_{n\to\infty}\left(\frac{3n^2+2n-1}{4n^2-5n+6}\right)^2=\left(\lim_{n\to\infty}\frac{3n^2+2n-1}{4n^2-5n+6}\right)^2=\left(\lim_{n\to\infty}\frac{3+\frac{2}{n}-\frac{1}{n^2}}{4-\frac{5}{n}+\frac{6}{n^2}}\right)^2=\left(\frac{3}{4}\right)^2=\frac{9}{16}.$$

$$\lim_{n\to\infty}\left(2+\frac{3}{n}\right)\cdot\left(3-\frac{4}{n}\right)^2\cdot\left(\frac{4}{n^2}-1\right)=\lim_{n\to\infty}\left(2+\frac{3}{n}\right)\cdot\lim_{n\to\infty}\left(3-\frac{4}{n}\right)^2\cdot\lim_{n\to\infty}\left(\frac{4}{n^2}-1\right)=$$

$$=2\cdot\left(\lim_{n\to\infty}\left(3-\frac{4}{n}\right)\right)^2\cdot(-1)=-2\cdot 9=-18.$$

$$\lim_{n\to\infty}2^{\frac{10n+9}{2n-8}}=2^{\lim\limits_{n\to\infty}\frac{10n+9}{2n-8}}=2^{\frac{10}{2}}=2^5=32.$$

$$\lim_{n\to\infty}\lg\frac{2n^2-5}{3n^2-6}=\lg\left(\lim_{n\to\infty}\frac{2n^2-5}{3n^2-6}\right)=\lg\frac{2}{3}=\lg 2-\lg 3.$$

Formula $\lim\limits_{n\to\infty}\sqrt[n]{n}=1$ and the consequence of it $\lim\limits_{n\to\infty}\sqrt[n]{\frac{1}{n}}=1,$ as well as $\lim\limits_{n\to\infty}\sqrt[n]{a}=1,\ a>1$ may be beneficial when doing the following exercises.

$$\lim_{n\to\infty}\sqrt[n]{5\cdot n}=\lim_{n\to\infty}\sqrt[n]{5}\cdot\lim_{n\to\infty}\sqrt[n]{n}=1\cdot 1=1.$$

$$\lim_{n\to\infty}\sqrt[n]{n^4}=\lim_{n\to\infty}\left(\sqrt[n]{n}\right)^4=\left(\lim_{n\to\infty}\sqrt[n]{n}\right)^4=1^4=1.$$

$$\lim_{n\to\infty}\frac{\log_a n}{n}=\lim_{n\to\infty}\frac{1}{n}\cdot\lim_{n\to\infty}\log_a n=0\cdot\log_a\lim_{n\to\infty}=0\cdot\infty=0.$$

Very rarely, but it is used to calculate boundaries, such a theorem: if three variables x_n, y_n, z_n related by the relationship $x_n\le z_n\le y_n$ and x_n and y_n have equal boundaries, then the same boundary has and z_n.

To find $\lim\limits_{n\to\infty}\sqrt[n]{3n+2}.$

Solution:

Let be $n>2,$ then $\sqrt[n]{n}<\sqrt[n]{3n+2}<\sqrt[4]{4n}.$

$\lim\limits_{n\to\infty}\sqrt[n]{n}=1,\ \lim\limits_{n\to\infty}\sqrt[4]{4n}=\lim\limits_{n\to\infty}\sqrt[n]{4}\cdot\lim\limits_{n\to\infty}\sqrt[n]{n}=1\cdot1=1.$

Well then, $\lim\limits_{n\to\infty}\sqrt[n]{3n+2}=1.$

Answer: 1.

Transferring irrationality to the denominator, which is achieved by multiplying and dividing by the expression, associated with the given, becomes useful when solving this type of exercise:

$$\lim_{n\to\infty}\left(\sqrt{n+3}-\sqrt{n}\right)=\lim_{n\to\infty}\frac{\left(\sqrt{n+3}-\sqrt{n}\right)\left(\sqrt{n+3}+\sqrt{n}\right)}{\sqrt{n+3}+\sqrt{n}}=\lim_{n\to\infty}\frac{n+3-n}{\sqrt{n+3}+\sqrt{n}}=\lim_{n\to\infty}\frac{3}{\sqrt{n+3}+\sqrt{n}}=$$

$$=3\cdot\lim_{n\to\infty}\frac{1}{\sqrt{n+3}+\sqrt{n}}=3\cdot\frac{1}{\infty}=0.$$

$$\lim_{n\to\infty}\left(\sqrt{3n+2}-\sqrt{n-1}\right)=\lim_{n\to\infty}\frac{\left(\sqrt{3n+2}-\sqrt{n-1}\right)\cdot\left(\sqrt{3n+2}+\sqrt{n-1}\right)}{\sqrt{3n+2}+\sqrt{n-1}}=$$

$$=\lim_{n\to\infty}\frac{3n+2-n+1}{\sqrt{3n+2}+\sqrt{n-1}}=\lim_{n\to\infty}\frac{2n+3}{\sqrt{3n+2}+\sqrt{n-1}}=\lim_{n\to\infty}\frac{2+\frac{3}{n}}{\sqrt{\frac{3n}{n^2}+\frac{2}{n^2}}+\sqrt{\frac{n}{n^2}-\frac{1}{n^2}}}=$$

$$=\lim_{n\to\infty}\frac{2+\frac{3}{n}}{\sqrt{\frac{3}{n}+\frac{2}{n^2}}+\sqrt{\frac{1}{n}-\frac{1}{n^2}}}=\frac{2+0}{\sqrt{0+0}+\sqrt{0-0}}=\frac{2}{0}=\infty.$$

Answer: ∞.

$$\lim_{n\to\infty}\sqrt{n}\cdot\left(\sqrt{n+1}-\sqrt{n}\right)=\lim_{n\to\infty}\left(\sqrt{n^2+n}-n\right)=\lim_{n\to\infty}\frac{\left(\sqrt{n^2+n}-n\right)\left(\sqrt{n^2+n}+n\right)}{\sqrt{n^2+n}+n}=$$

$$=\lim_{n\to\infty}\frac{n^2+n-n^2}{\sqrt{n^2+n}+n}=\lim_{n\to\infty}\frac{\frac{n}{n}}{\sqrt{\frac{n^2}{n^2}+\frac{1}{n}}+1}=\frac{1}{\sqrt{1+0}+1}=\frac{1}{1+1}=\frac{1}{2};$$

$$\lim_{n\to\infty}\frac{\sqrt{n^2+1}}{n+1}=\lim_{n\to\infty}\frac{\sqrt{\frac{n^2}{n^2}+\frac{1}{n^2}}}{1+\frac{1}{n}}=\frac{1+0}{1+0}=1.$$

$$\lim_{n\to\infty}\sqrt[3]{\frac{n^2-n+1}{8n^2+n+3}}=\sqrt[3]{\lim_{n\to\infty}\frac{n^2-n+1}{8n^2+n+3}}=\sqrt[3]{\frac{1}{8}}=\frac{1}{2};$$

$$\lim_{n\to\infty}\left(\sqrt[3]{\frac{5n}{4n+3}}\right)^{-\frac{1}{2}}=\lim_{n\to\infty}\left(\frac{5n}{4n+3}\right)^{-\frac{1}{6}}=\left(\frac{5}{4}\right)^{-\frac{1}{6}}=\frac{1}{\left(\frac{5}{4}\right)^{-\frac{1}{6}}}=\left(\frac{4}{5}\right)^{\frac{1}{6}}=\sqrt[6]{\frac{4}{5}}.$$

$$\lim_{n\to\infty}\frac{1-2+3-4+5-6+\ldots-2n}{\sqrt{n^2+2}+\sqrt[3]{3n^2-1}}=\lim_{n\to\infty}\frac{(1+3+5+7+\ldots+2n-1)-(2+4+6+\ldots+2n)}{\sqrt{n^2+2}+\sqrt[3]{3n^2-1}}=$$

$$=\lim_{n\to\infty}\frac{\frac{1+2n-1}{2}\cdot n-\frac{2+2n}{2}\cdot n}{\sqrt{n^2+2}+\sqrt[3]{3n^2-1}}=\lim_{n\to\infty}\frac{2n-n-n^2}{\sqrt{n^2+2}+\sqrt[3]{3n^2-1}}=\lim_{n\to\infty}\frac{n-n^2}{\sqrt{n^2+2}+\sqrt[3]{3n^2-1}}=-\frac{1}{3}.$$

$$\lim_{n\to\infty}\left(\sqrt[3]{1-n^3}+n\right)=\lim_{n\to\infty}\frac{\left(\sqrt[3]{1-n^3}+n\right)\cdot\left(\left(\sqrt[3]{1-n^3}\right)^2-n\sqrt[3]{1-n^3}+n^2\right)}{\left(\sqrt[3]{1-n^3}\right)^2-n\sqrt[3]{1-n^3}+n^2}=\left\{\sqrt[3]{1-n^3}=a,\ n=b\right\}=$$

$$=\lim_{n\to\infty}\frac{1-n^3+n^3}{\sqrt[3]{1-n^3}-\sqrt[n]{1-n^3}\cdot n+n^2}=\lim_{n\to\infty}\frac{1}{\infty}=0.$$

Limit of function

To find the limit of an entire rational function, it is necessary to replace the argument with its limit value.

To find the limit of a fractional-rational function, it is necessary to replace the argument with its limit value, provided that the denominator does not vanish in this case.

$$\lim_{x\to 2}\left(x^3-\frac{3}{4}x^2+2x-5\right)=2^3-\frac{3}{4}\cdot 2^2+2\cdot 2-5=8-3+4-5=4.$$

When calculating the limits of fractional-rational functions, it is necessary first to check whether the denominator does not vanish when the argument is replaced by its limiting value.

$$\lim_{x\to 3}\frac{x+x+2}{3x^2+2x+8}=\frac{3^2+3+2}{3^2+2\cdot 3+8}=\frac{9+5}{9+14}=\frac{14}{23}.$$

$x^2+2x+8=3^2+2\cdot 3+8=$ $=9+6+8=23\neq 0.$	$\lim_{x\to 2}\frac{x^3-8}{x-2}=\lim_{x\to 2}\frac{(x-2)(x^2+2x+4)}{x-2}=\lim_{x\to 2}(x^2+2x+4)=$
When $x=2$ $x^3-8=2^3-8=0$ i $x-2=2-2=0.$	$=2^2+2\cdot 2+4=12.$

$$\lim_{x\to 1}\frac{x^m-1}{x^n-1}=\lim_{x\to 1}\frac{(x-1)\cdot\left(x^{m-1}+^{m-2}+x^{m-3}+\ldots+x+1\right)}{(x-1)\cdot\left(x^{n-1}+^{n-2}+x^{n-3}+\ldots+x+1\right)}=$$

x^m-1 | $x-1$ x^m-x^{m-1} | $x^{m-1}+x^{m-2}+x^{m-3}+\ldots+x+1$ $x^{m-1}-1$ $x^{m-1}-x^{m-2}$ $x^{m-2}-x^{m-3}$ $x^{m-2}-x^2$ x^2-1 x^2-x $x-1$ $x-1$ 0	$=\lim_{x\to 1}\frac{x^{m-1}+^{m-2}+x^{m-3}+\ldots+x+1}{x^{n-1}+^{n-2}+x^{n-3}+\ldots+x+1}=\frac{\overbrace{1+1+1+\ldots+1+1}^{m\ \text{разів}}}{\underbrace{1+1+1+\ldots+1+1}_{n\ \text{разів}}}=\frac{m}{n}.$

$$\lim_{x\to -1}\frac{1+\sqrt[7]{x}}{1+\sqrt[5]{x}}=\lim_{y\to -1}\frac{1+\sqrt[7]{y^{35}}}{1+\sqrt[5]{y^{35}}}=\lim_{y\to -1}\frac{1+y^5}{1+y^7}=$$

Substitution $x=y^{35}$

$1+y^5 \;\big|\; 1+y$ $\underline{y^4+y^5} \;\big|\; y^4-y^3+y^2-y+1$ $1-y^4$ $\underline{y^3-y^4}$ $1+y^3$ $\underline{y^2+y^3}$ $1-y^2$ $\underline{-y-y^2}$ $1+y$ $\underline{1+y}$ 0	$=\lim_{y\to -1}\frac{(1+y)(y^4-y^3+y^2-y+1)}{(1+y)(y^6-y^5+y^4-y^3+y^2-y+1)}=$ $=\lim_{y\to -1}\frac{(-1)^4-(-1)^3+(-1)^2-(-1)+1}{(-1)^6-(-1)^5+(-1)^4-(-1)^3+(-1)^2-(-1)+1}=\frac{5}{7}.$

Answer: $\frac{5}{7}$.

$$\lim_{x\to 3}\frac{2x-5}{x^2-7x+12}=\frac{2\cdot 3-5}{3^2-7\cdot 3+12}=\frac{1}{0}=\infty.$$

$$\lim_{x\to\infty}\frac{2x^3+x^2+5}{3x^3+x-1}=\lim_{x\to\infty}\frac{2+\frac{1}{x}+\frac{5}{x^3}}{3+\frac{1}{x^2}-\frac{1}{x^3}}=\frac{2}{3}.$$

If the exponent is a constant value and the limit of the exponent exists, then you can go to the limits at the base of the exponent, that is

$$\lim_{x\to a}(f(x))^k=\left(\lim_{x\to a}f(x)\right)^k.$$

$$\lim_{x\to 27}\sqrt[3]{x}=\sqrt[3]{\lim_{x\to 27}x}=\sqrt[3]{27}=3.$$

$$\lim_{x\to 2}\sqrt{2x^2+7}=\sqrt{\lim_{x\to 2}(2\cdot 2^2+7)}=\sqrt{15}.$$

$$\lim_{x\to 0}\frac{\sqrt{1+x}-1}{x}=\lim_{x\to 0}\frac{(\sqrt{1+x}-1)\cdot(\sqrt{1+x}+1)}{x(\sqrt{1+x}+1)}=\lim_{x\to 0}\frac{1+x-1}{x(\sqrt{1+x}+1)}=\lim_{x\to 0}\frac{1}{\sqrt{1+x}+1}=\frac{1}{1+1}=\frac{1}{2}.$$

In uncertainties of the kind $\frac{0}{0}$ you can transfer irrationality from the numerator to the denominator, or vice versa.

$$\lim_{x\to 2}\frac{\sqrt{x^2+5}-3}{x-2}=\lim_{x\to 2}\frac{\left(\sqrt{x^2+5}-3\right)\cdot\left(\sqrt{x^2+5}+3\right)}{(x-2)\cdot\left(\sqrt{x^2+5}+3\right)}=\lim_{x\to 2}\frac{x^2+5-9}{(x-2)\cdot\left(\sqrt{x^2+5}+3\right)}=$$

$$=\lim_{x\to 2}\frac{x^2-4}{(x-2)\cdot\left(\sqrt{x^2+5}+3\right)}=\lim_{x\to 2}\frac{(x-2)\cdot(x+2)}{(x-2)\cdot\left(\sqrt{x^2+5}+3\right)}=\lim_{x\to 2}\frac{x+2}{\sqrt{x^2+5}+3}=\lim_{x\to 2}\frac{2+2}{\sqrt{2^2+5}+3}=$$

$$=\frac{4}{6}=\frac{2}{3}.$$

$$\lim_{x\to 3}\frac{\sqrt{3x+7}-\sqrt{2x+10}}{\sqrt{4x+13}-\sqrt{x+22}}=\lim_{x\to 3}\frac{\left(\sqrt{3x+7}-\sqrt{2x+10}\right)\cdot\left(\sqrt{3x+7}+\sqrt{2x+10}\right)}{\left(\sqrt{4x+13}-\sqrt{x+22}\right)\cdot\left(\sqrt{4x+13}+\sqrt{x+22}\right)}=$$

$$=\lim_{x\to 3}\frac{\left(\sqrt{4x+13}+\sqrt{x+22}\right)}{\left(\sqrt{3x+7}+\sqrt{2x+10}\right)}=\lim_{x\to 3}\frac{(3x+7-2x-10)\left(\sqrt{4x+13}+\sqrt{x+22}\right)}{(4x+13-x-22)\left(\sqrt{3x+7}+\sqrt{2x+10}\right)}=$$

$$=\lim_{x\to 3}\frac{(x-3)\left(\sqrt{4x+13}+\sqrt{x+22}\right)}{(3x-9)\left(\sqrt{3x+7}+\sqrt{2x+10}\right)}=\lim_{x\to 3}\frac{(x-3)\left(\sqrt{4x+13}+\sqrt{x+22}\right)}{3\cdot(x-3)\left(\sqrt{3x+7}+\sqrt{2x+10}\right)}=$$

$$=\lim_{x\to 3}\frac{\sqrt{4x+13}+\sqrt{x+22}}{3\cdot\left(\sqrt{3x+7}+\sqrt{2x+10}\right)}=\frac{\sqrt{4x+13}+\sqrt{x+22}}{3\cdot\left(\sqrt{3\cdot 3+7}+\sqrt{2\cdot 3+10}\right)}=\frac{5+5}{12+12}=\frac{10}{24}=\frac{5}{12}.$$

$$\lim_{x\to 7}\frac{\sqrt[3]{x-6}-1}{x-7}=\lim_{x\to 7}\frac{\left(\sqrt[3]{x-6}-1\right)\cdot\left(\left(\sqrt[3]{x-6}\right)^2+\sqrt[3]{x-6}\cdot 1+1^2\right)}{\left(\left(\sqrt[3]{x-6}\right)^2+\sqrt[3]{x-6}+1\right)(x-7)}=$$

$$=\lim_{x\to 7}\frac{x-6-1}{\left(\left(\sqrt[3]{x-6}\right)^2+\sqrt[3]{x-6}+1\right)(x-7)}=\lim_{x\to 7}\frac{x-7}{\left(\left(\sqrt[3]{x-6}\right)^2+\sqrt[3]{x-6}+1\right)(x-7)}=$$

$$=\lim_{x\to 7}\frac{1}{\left(\sqrt[3]{x-6}\right)^2+\sqrt[3]{x-6}+1}=\frac{1}{\left(\sqrt[3]{1}\right)^2+\sqrt[3]{1}+1}=\frac{1}{3}.$$

If the limit value of the argument with trigonometric functions belongs to the domain of definition, then the value of the argument can be replaced by its limit value.

$$\lim_{x\to a}\sin x=\sin a;\ \lim_{x\to a}\cos x=\cos a;\ \lim_{x\to a}tgx=tga;\ \lim_{x\to a}ctgx=ctga;$$

$$\lim_{x\to\frac{\pi}{2}}\frac{1-\sin x}{\cos^2 x}=\lim_{x\to\frac{\pi}{2}}\frac{1-\sin x}{1-\sin^2 x}=\lim_{x\to\frac{\pi}{2}}\frac{1-\sin x}{(1-\sin x)(1+\sin x)}=\lim_{x\to\frac{\pi}{2}}\frac{1}{1+\sin x}=\frac{1}{1+1}=\frac{1}{2}.$$

The first wonderful limit: $\lim_{x\to 0}\frac{\sin kx}{x}=1.$

Calculate: $\lim_{x\to 0}\frac{\sin kx}{x}=\lim_{y\to 0}\frac{\sin y}{\frac{y}{k}}=\lim_{x\to 0}\frac{k\sin y}{y}=k\cdot 1=k.$

We denote: $kx=y,\ x=\frac{y}{k}.$ If $x\to 0,$ то $y\to 0.$

$$\lim_{x\to 0}\frac{\sin kx}{\sin lx}=\lim_{x\to 0}\frac{\frac{\sin kx}{x}}{\frac{\sin lx}{kx}}=\frac{k}{l};$$

$$\lim_{x\to 0}\frac{tgkx}{x}=\lim_{x\to 0}\left(\frac{1}{x}\cdot\frac{\sin kx}{\cos kx}\right)=\lim_{x\to 0}\left(\frac{\sin kx}{x}\cdot\frac{1}{\cos kx}\right)=\lim_{x\to 0}\frac{\sin kx}{x}\cdot\lim_{x\to 0}\frac{1}{\cos kx}=k\cdot\frac{1}{1}=k.$$

$$\lim_{x\to 0}\frac{1-\cos mx}{x^2}=\lim_{x\to 0}\frac{2\sin^2\frac{mx}{2}}{x^2}=2\lim_{x\to 0}\frac{\sin\frac{mx}{2}}{x}\cdot\lim_{x\to 0}\frac{\sin\frac{mx}{2}}{x}=2\lim_{x\to 0}\frac{\sin\frac{mx}{2}\cdot x}{x}\cdot\lim_{x\to 0}\frac{\sin\frac{mx}{2}\cdot x}{x}=$$

$$=2\cdot\frac{m}{2}\cdot\frac{m}{2}=\frac{m^2}{2}.$$

$$\lim_{x\to 0}x\cdot ctgx=\lim_{x\to 0}\left(x\cdot\frac{\cos x}{\sin x}\right)=\lim_{x\to 0}\frac{x}{\sin x}\cdot\lim_{x\to 0}\cos x=1\cdot 1=1.$$

Second wonderful limit: $\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^x=e.$ $\lim\limits_{x\to 0}\left(1+\frac{k}{x}\right)^x=e^k.$ $\lim\limits_{x\to 0}(1+x)^{\frac{1}{x}}=e.$

$$\lim_{x\to 0}(1+kx)^{\frac{1}{x}}=e^k.$$

Tip: If the base of the exponent is a constant value, then you can go to the border in the exponent.

$$\lim_{x\to 2}4^{\frac{2x}{x+1}}=4^{\lim\limits_{x\to 2}\frac{2x}{x+1}}=4^{\frac{2\cdot 2}{2+1}}=4^{\frac{4}{3}}=4\sqrt[3]{4}.$$

$$\lim_{x\to 2}a^{\frac{\sqrt{2+x}-2}{x-2}}=a^{\lim\limits_{x\to 2}\frac{\sqrt{2+x}-2}{x-2}}=a^{\lim\limits_{x\to 2}\frac{(\sqrt{2+x}-2)(\sqrt{2+x}+2)}{(x-2)(\sqrt{2+x}+2)}}=a^{\lim\limits_{x\to 2}\frac{2+x-4}{(x-2)(\sqrt{2+x}+x)}}=a^{\lim\limits_{x\to 2}\frac{x-2}{(x-2)(\sqrt{2+x}+2)}}=a^{\lim\limits_{x\to 2}\frac{1}{\sqrt{2+x}+2}}=$$

$$=a^{\frac{1}{\sqrt{2+2}+2}}=a^{\frac{1}{4}}=\sqrt[4]{a}.$$

$$\lim_{x\to\infty}\left(1-\frac{1}{x}\right)^x=\lim_{x\to\infty}\left(1+(-1)\cdot\frac{1}{x}\right)^x=e^{-1}=\frac{1}{e}.$$

$$\lim_{x\to\infty}\left(1+\frac{2}{3x}\right)^x=\lim_{x\to\infty}\left(1+\frac{2}{3}\cdot\frac{1}{x}\right)^x=e^{\frac{2}{3}}.$$

$$\lim_{x\to 0}\left(1+5tg^2x\right)^{3ctg^2x}=\lim_{x\to 0}\left(1+5tg^2x\right)^{\frac{3}{tg^2x}}=\lim_{x\to 0}\left(\left(1+tg^2x\right)^{\frac{1}{tg^2x}}\right)^3=\left(e^5\right)^3=e^{15}.$$

Self-study assignments:

$\lim\limits_{n\to\infty}\frac{3n^3+2n^2-3n+2}{5n^3-7n+20}.$ Answer: $\frac{3}{5}.$

$\lim\limits_{n\to\infty}\frac{4n^2-6n+5}{n^3-6n+5}.$ Answer: $0.$

$\lim\limits_{n\to\infty}\frac{3n+5}{4-6n}.$ Answer: $-\frac{1}{2}.$

$\lim\limits_{n\to\infty}\frac{6n^2-5n+4}{3n^2-2n+1}.$ Answer: $2.$

$\lim\limits_{n\to\infty}\left(\frac{2n^3-3n^2+4n+5}{5n^3-6n^2+7n+8}\right)^3$ Answer: $\frac{8}{125}.$

$\lim\limits_{n\to\infty}\left(\frac{1}{3}\right)^{\frac{2n+1}{n-7}}.$ Answer: $\frac{1}{9}.$

$\lim\limits_{n\to\infty}\frac{2n^3+3}{4n^2-5}.$ Answer: $\infty.$

$\lim\limits_{n\to\infty}\frac{6n+7}{8n^2+9n-10}.$ Answer: $0.$

$\lim 5^{\frac{4n+3}{2n-5}}.$ Answer: $25.$

$\lim\limits_{n\to\infty}6^{\frac{n+5}{n^2-5}}.$ Answer: $1.$

$\lim\limits_{n\to\infty}\left(\log_a\frac{7n}{14n+22}\right).$ Answer: $-\log_a 2.$

$\lim\limits_{n\to\infty}\sqrt[m]{6n}.$ Answer: $1.$

$\lim\limits_{n\to\infty}\sqrt[n]{5n+3}.$ Answer: $1.$

$\lim\limits_{n\to\infty}\frac{\sqrt{n^2+4n}}{\sqrt[3]{n^3-3n^2}}.$ Answer: $1.$

$\lim\limits_{n\to\infty}\frac{\sqrt[3]{n^2+6n}}{2n+3}.$ Answer: $0.$

$\lim\limits_{x\to 2}\left(2x^3-7x^2+4x+2\right).$ Answer: $-2.$

$\lim\limits_{x\to 1}\frac{x^2-3x+2}{x^2+x+4}.$ Answer: $0.$

$\lim\limits_{x\to 3}\frac{x^2+x-12}{2x^2-9x+9}.$ Answer: $2\frac{1}{3}.$

$\lim\limits_{x\to 0}\frac{5x^3-6x^2}{4x^5+2x^3+x^2}.$ Answer: $-6.$

$\lim\limits_{n\to\infty}\sqrt[4]{n^9}.$ Answer: $1.$

$\lim\limits_{n\to\infty}\sqrt[4]{\frac{1}{2}(10n+2)}.$ Answer: $1.$

$\lim\limits_{n\to\infty}\left(\sqrt{n^2+n+1}-\sqrt{n^2-n+1}\right)$ Answer: $1.$

$\lim\limits_{n\to\infty}\left((n+1)^{\frac{2}{3}}-(n-1)^{\frac{2}{3}}\right).$ Answer: $0.$

$\lim\limits_{x\to 4}\left(\frac{1}{2}x^3-x+2\right).$ Answer: $30.$

$\lim\limits_{x\to -1}\frac{x^2-x+1}{2x^3-x^2+x+2}.$ Answer: $-\frac{3}{2}.$

$\lim\limits_{x\to 1}\frac{x^3-x^2-x+1}{x^3-3x+2}.$ Answer: $-\frac{2}{3}.$

$\lim\limits_{x\to -3}\frac{x^3+5x^2+3x-9}{x^3-3x^2-45x-81}.$ Answer: $\frac{1}{3}.$

$\lim\limits_{x\to 1}\frac{1-\sqrt[4]{x}}{1-\sqrt[6]{x}}.$ Answer: $\frac{3}{2}.$

$\lim\limits_{x\to 1}\frac{1-\sqrt[3]{x}}{1-\sqrt[5]{x}}.$ Answer: $\frac{5}{3}.$

$\lim\limits_{x\to1}\dfrac{x^2+x+1}{x^4-3x+2}.$ Answer: ∞.

$\lim\limits_{x\to2}\left(\dfrac{1}{x-2}-\dfrac{1}{x^2-4}\right).$ Answer: ∞.

$\lim\limits_{x\to1}\left(\dfrac{1}{1-x}-\dfrac{3}{1-x^2}\right).$ Answer: -1.

$\lim\limits_{x\to\infty}\dfrac{5x^4-7x^2+5x-4}{x^4+x^2+x+1}.$ Answer: 5.

$\lim\limits_{x\to\infty}\dfrac{(x+3)(x+4)(x+5)}{x^4+x-1}.$ Answer: 0.

$\lim\limits_{x\to\infty}\dfrac{x^3+x+2}{x^2-x+1}.$ Answer: ∞.

$\lim\limits_{x\to\infty}\dfrac{3x^3-2x^2+x}{2x^2+x-1}.$ Answer: $\dfrac{3}{2}$.

$\lim\limits_{x\to\infty}\dfrac{7x^4+2x^3-14}{5x^4+x^3+x^2+x-1}.$ Answer: 1,4.

$\lim\limits_{x\to\infty}\left(\dfrac{x^3}{5x^2+1}-\dfrac{3x^2}{15x+1}\right).$ Answer: $\dfrac{1}{75}$.

$\lim\limits_{x\to\infty}\left(\dfrac{x^2+x-1}{2x^2-x+1}\right)^2.$ Answer: $\dfrac{1}{8}$.

$\lim\limits_{x\to\infty}\left(\dfrac{5x^2-x}{x^2-3}-\dfrac{3x^3-4}{x^3-x}\right)^4.$ Answer: 16.

$\lim\limits_{x\to-243}\sqrt[5]{x}.$ Answer: -3.

$\lim\limits_{x\to3}\dfrac{\sqrt{x+6}-3}{x-3}.$ Answer: $\dfrac{1}{6}$.

$\lim\limits_{x\to-2}\dfrac{\sqrt{6+x}-2}{x+2}.$ Answer: $\dfrac{1}{4}$.

$\lim\limits_{x\to2}\dfrac{\sqrt{x^2+21}-5}{x-2}.$ Answer: $\dfrac{2}{5}$.

$\lim\limits_{x\to1}\dfrac{\sqrt{5-x^2}-2}{1-x}.$ Answer: $\dfrac{1}{2}$.

$\lim\limits_{x\to3}\dfrac{\sqrt{x+8}-\sqrt{8x+1}}{\sqrt{5-x}-\sqrt{7x-3}}.$ Answer: $\dfrac{7}{12}$.

$\lim\limits_{x\to-3}\dfrac{\sqrt{x^2+7}-\sqrt{7-3x}}{\sqrt{x+3}-\sqrt{x^2-9}}.$ Answer: 0.

$\lim\limits_{x\to4}\dfrac{\sqrt{x+5}-\sqrt{3x^2-39}}{\sqrt{x^2-3}-\sqrt{2x^2-19}}.$ Answer: $\dfrac{23\sqrt{13}}{24}$.

$\lim\limits_{x\to-2}\dfrac{x+2}{\sqrt{x^2+5}-3}.$ Answer: $-\dfrac{3}{2}$.

$$\lim_{x\to 0}\frac{\sqrt[3]{1+x}-\sqrt[3]{3x+1}}{6x}.$$ Answer: $-\frac{1}{9}.$

$$\lim_{x\to 1}\frac{\sqrt[3]{2x-1}-\sqrt[3]{3x-3}}{\sqrt{4x-3}-1}.$$ Answer: $-\frac{1}{6}.$

$$\lim_{x\to 0}\frac{7x}{\sqrt[3]{6x-1}+\sqrt[3]{2x+1}}.$$ Answer: $\frac{21}{8}.$

$$\lim_{x\to \frac{\pi}{4}}\frac{\cos x-\sin x}{1-tg^2x}.$$ Answer: $\frac{1}{2\sqrt{2}}.$

$$\lim_{x\to \frac{\pi}{6}}\frac{1-4\sin^2 x}{\cos 3x}.$$ Answer: $-\frac{2\sqrt{3}}{3}.$

$$\lim_{x\to 0}\frac{\sin 5x}{\sin 7x}.$$ Answer: $\frac{5}{7}.$

$$\lim_{x\to 0}\frac{tgkx}{tgex}.$$ Answer: $\frac{k}{e}.$

$$\lim_{x\to 0}\frac{\sin^2 \alpha x}{x^2}.$$ Answer: $\alpha^2.$

$$\lim_{x\to \frac{\pi}{3}}\frac{\sin\left(x-\frac{\pi}{3}\right)}{1-2\cos x}.$$ Answer: $\frac{1}{\sqrt{3}}.$

$$\lim_{x\to 0}\frac{tgx-\sin x}{x^3}.$$ Answer: $\frac{1}{2}.$

$$\lim_{x\to \infty}2^{\frac{3x}{x+2}}.$$ Answer: $8.$

$$\lim_{x\to \infty}\left(1-\frac{k}{x}\right)^x.$$ Answer: $e^{-k}=\frac{1}{e^k}.$

$$\lim_{x\to 1}\frac{x^3-1}{x^2+5x-6}.$$ Answer: $\frac{3}{7}.$

$$\lim_{x\to 7}\frac{2-\sqrt{x-3}}{x^2-49}.$$ Answer: $-\frac{1}{56}.$