Section 21
Arithmetic progression

A numerical sequence is a renumbered series of digits arranged in ascending order of their place numbers. The numbers in the sequence are called - its members.
An arithmetic progression is a numerical sequence, each term of which, starting from the second, is equal to the previous one, to which it is attached - the same number *d*.
Convenient to denote arithmetic progression $\div$
$a_n = a_{n-1} + a,\ \ d = a_n - a_{n-1} -$ difference of arithmetic progression.
To define an arithmetic progression, it is enough to indicate its first term a_1 and the difference *d*.
A characteristic property of arithmetic progression: each member, starting with the second is the average of the preceding and following members, that is,

$$a_n = \frac{a_{n-1} + a_{n+1}}{2}, n \in Z.$$

Formula n-th member:

$$a_n = a_1 + d \cdot (n-1).$$

Formula for the sum of *n* terms of an arithmetic progression:

$$S_n = \frac{a_1 + a_n}{2} \cdot n;\ S_n = \frac{2a_1 + d \cdot (n-1)}{2} \cdot n.$$

A very valuable property of the members of an arithmetic progression: the sums of the members equidistant from the ends of the arithmetic progression are equal to each other, that is: $a_1 + a_n = a_2 + a_{n-1}$.
Given: $a_1 = 3, a_2 = 7.$
To find: a_{15} i S_{15}.

Solution:

Find the difference in progression: $d = a_2 - a_1,\quad d = 7 - 3 = 4.$
From the requirement of the problem it follows that $n = 15$.
By the formula of the *n*-th members $\div$ $a_n = a_1 + d(n-1)$ we have:
$a_{15} = 3 + 4 \cdot (15 - 1) = 3 + 56 = 59.$

$$S_n = \frac{a_1 + a_n}{2} \cdot n,\ S_{15} = \frac{3 + 59}{2} \cdot 15 = 31 \cdot 15 = 465.$$

Answer: $59; 465.$
Given: $a_1 = -5;\ d = 4.$ To find a_{10} and S_{100}.

Solution:

$a_{11} = a_1 + d(n-1),\ a_{10} = -5 + 4 \cdot (10 - 1) = -5 + 36 = 31.$

$$S_n = \frac{2a_1 + d(n-1)}{2} \cdot n;$$

$$S_{100} = \frac{2 \cdot (-5) + 4 \cdot (100 - 1)}{2} \cdot 100 = \frac{-10 + 396}{2} \cdot 100 = \frac{386}{2} \cdot 100 = 193 \cdot 100 = 19300.$$

Answer: 31; 19300.

Given: $\div S_7 105, d=4$. To find a_1 и a_7.

Solution:

$a_n = a_1 + d(n-1)$.

Apply it up to the seventh member $\div$

$a_7 = a_1 + 4\cdot(n-1);\ a_7 = a_1 + 24.$

$S_n = \frac{a_1 + a_n}{2}\cdot n;\ S_7 = \frac{a_1 + a_7}{2}\cdot 7;\ 105 = \frac{a_1 + a_1 + 24}{2}\cdot 7;\ 105 = \frac{2a_1 + 24}{2}\cdot 7;\ 105 = (a_1 + 12)\cdot 7;$

$a_1 + 12 = \frac{105}{7};\ a_1 + 12 = 15;\ a_1 = 15 - 12;\ a_1 = 3.$

$a_7 = 3 + 24;\ a_7 = 27.$

Answer: 3; 27.

Given $\div\ d = -3,\ a_n = 2,\ S_n = 57$. To find a_1 and n.

Solution:

$a_n = a_1 + d\cdot(n-1);\ 2 = a_n - 3\cdot(n-1);\ 2 = a_1 - 3n + 3;\ a_1 - 3n = -1\ (1).$

$S_n = \frac{a_1 + a_n}{2}\cdot n;\ 57 = \frac{a_1 + 2}{2}\cdot n;\ 114 = (a_1 + 2)\cdot n\ (2)$. Insofar as a_1 and n must satisfy the conditions of the given problem, then you can create such a system of equations:

$\begin{cases} a_1 - 3n = -1, \\ (a_1 + 2)\cdot n = 114. \end{cases} \begin{cases} a_1 = 3n - 1, \\ (3n - 1 + 2)\cdot n = 114. \end{cases} \begin{cases} a_1 = 3n - 1, \\ (3n + 1)\cdot n = 114. \end{cases}$

$3n^2 + n - 114 = 0,\ D = 1 + 12\cdot 114 = 1369 = 37^2.$

$n_1 = \frac{-1 - 37}{6} = -\frac{38}{6} -$ does not satisfy the condition of the problem, because $n \in Z$.

$n_2 = \frac{-1 + 37}{6} = \frac{36}{6} = 6 \in N.\ a_1 = 3\cdot 6 - 1 = 17.$

Answer: 6; 17.

Between the numbers 17 and 32, insert five such numbers so that they, together with these numbers, form an arithmetic progression.

Solution:

Let the $a_1 = 17,\ a_3, a_4, a_5, a_6, a_7 = 32.$

According to the formula $a_n = a_1 + d\cdot(n-1)$ we find d:

$17 + d\cdot(7-1) = 32,\ 6d = 32 - 17,\ 6d = 15,\ d = \frac{15}{6} = 2{,}6.$

$a_2 = 17 + 2{,}5 = 19{,}5,\ a_3 = 19{,}5 + 2{,}5 = 22,\ a_4 = 22 + 2{,}5 = 24{,}5,\ a_5 = 24{,}5 + 2{,}5 = 27,$

$a_6 = 27 + 2{,}5 = 29{,}5.$

Answer: *19,5; 22; 24,5; 27; 29,5.*

Given $\div\ a_6 + a_9 + a_{12} + a_{15} = 20$. To find S_{20}.

Solution:

Members of a_6 and a_{15} - equidistant from ends $\div$, comprising 20 members. Similarly, one can assert about a_9 and a_{12}. In this way, $a_6 + a_{15} = a_9 + a_{12}$,

$a_6 + a_9 + a_{12} + a_{15} = (a_6 + a_{15}) + (a_9 + a_{12}) = (a_6 + a_{15})\cdot 2 = 20;\ a_6 + a_{15} = \frac{20}{2} = 10.$

$a_1 + a_{20} = a_6 + a_{15} = 10.$ We find S_{20}: $S_{20} = \frac{a_1 + a_{20}}{2} \cdot 20 = \frac{10}{2} \cdot 20 = 100.$

Answer: *100.*

Prove that the sequence given by the *n*-th term formula $a_n = 2n - 5,$ is an arithmetic progression.

Solution:

The previous member of this sequence $a_{n-1} = 2(n-1) - 5 = 2n - 2 - 5 = 2n - 7.$ Next member of the sequence $a_{n+1} = 2(n+1) - 5 = 2n + 2 - 5 = 2n - 3.$

Find the arithmetic mean of the previous and subsequent terms of this sequence:

$$\frac{2n - 7 + 2n - 3}{2} = \frac{4n - 10}{2} = \frac{4n}{2} - \frac{10}{2} = 2n - 5 = a_n.$$

The statement about the characteristic property of the progression is acceptable in some arithmetic progression, acceptable for the given sequence a_n, hence the sequence a_n – is an arithmetic progression.

Given: $\div a_1 + a_5 = 24,\ a_2 \cdot a_5 = 60.$ To find a_1 and d.

Solution:

a_1, a_2, a_3, a_5 must satisfy such equations in the condition of the problem, and therefore they must satisfy such a system of equations:

$$\begin{cases} a_1 + a_5 = 24, \\ a_2 \cdot a_3 = 60. \end{cases}$$

Using the formula of *n*-th member $\div$, express a_2, a_3, a_5 through a_1 and d:

$a_2 = a_1 + d,\ a_3 = a_1 + d + d = a_1 + 2d,\ a_5 = a_1 + 4d.$

$$\begin{cases} a_1 + a_1 + 4d = 24, \\ (a_1 + d) \cdot (a_1 + 2d) = 60. \end{cases} \begin{cases} a_1 + a_1 + 4d = 24, \left|:2\right. \\ (a_1 + d) \cdot (a_1 + 2d) = 60. \end{cases} \begin{cases} a_1 + 2d = 12, \\ (a_1 + d) \cdot 12 = 60. \end{cases} \begin{cases} a_1 + 2d = 12, \\ a_1 + d = 5. \end{cases}$$

$$\begin{cases} a_1 + d + d = 12, \\ a_1 + d = 5. \end{cases} \begin{cases} 5 + d = 12, \\ a_1 + d = 5. \end{cases} \begin{cases} d = 12 - 5, \\ a_1 + 7 = 25. \end{cases} \begin{cases} d = 7, \\ a_1 = -2. \end{cases}$$

Answer: $-2; 7.$

Find the sum of all two-digit numbers.

Solution:

We have $\div$ of $a_1 = 10,\ d = 1,\ a_n = 99.$

$a_n = a_1 + d(n-1),\ 10 + 1 \cdot (n-1) = 99,\ n - 1 = 99 - 10,\ n = 89 + 1,\ n = 90.$

$$S_n = \frac{2a_1 + d(n-1)}{2} \cdot n,\ S_{90} = \frac{2 \cdot 10 + 1 \cdot (90-1)}{2} \cdot 90 = \frac{20 + 89}{2} \cdot 90 = \frac{109}{2} \cdot 90 = 4905.$$

Answer: $4905.$

Solve the equation: $1 + 7 + 13 + ... + x = 280, x \in N.$

Solution:

Obviously, the terms of this equation are members of an arithmetic progression, therefore $7 - 1 = 6,\ 13 - 7 = 6,\ d = 6.$

$$a_n = a_1 + d(n-1),\ x = 1 + 6(n-1),\ x = 1 + 6n - 6,\ 6n = x + 5,\ n = \frac{x+5}{6}.$$

$S_n = \frac{a_1 + a_n}{2} \cdot n,\ S_n = \frac{1+x}{2} \cdot \frac{x+5}{6} = \frac{(1+x)(x+5)}{12},\ \frac{x+5+x^2+5x}{12} = 280,\ x^2 + 6x + 5 = 3360,$

$x^2 + 6x - 3355 = 0,\ D = 36 + 13420 = 13456 = 116^2.$

$x_1 = \frac{-6-116}{2} = \frac{-122}{2} = -61 -$ does not satisfy the condition;

$x_2 = \frac{-6+116}{2} = 55.$

Answer: 55.

Find a_1 and d arithmetic progression, sum n the first terms of which is given by the formula $S_n = 5n^2 + 6n.$

Solution:

Find the sum of first term $S_1 = 5 \cdot 1^2 + 6 \cdot 1 = 11.$

The value of the sum of one member of the progression coincides with the value of the first member, that is $a_1 = 11.$

Let's calculate the sum of two terms $S_2 = 5 \cdot 2^2 + 6 \cdot 2 = 32.$

$S_2 = a_1 + a_2;\ a_2 = S_2 - a_1;\ a_2 = 32 - 11 = 21.$

The difference in progression: $d_2 = a_2 - a_1;\ d = 21 - 11 = 10.$

Answer: $a_1 = 11;\ d = 10.$

Prove that the sum of the first n odd numbers of a natural series is equal to the square of their number.

Proof:

$S_n = 1 + 3 + 5 + 7 + ... + (2n-1).$

Insofar as $3 - 1 = 5 - 3 = 7 - 5 = ... = (2n-1) - (2n-3) = 2,$ then we have the sum of the terms of the arithmetic progression with $a_1 = 1,\ d = 2,\ a_n = 2n - 1$ and number of members n.

According to the formula $S_n = \frac{a_1 + a_n}{2} \cdot n$ we have:

$S_n = \frac{1 + 2n - 1}{2} \cdot n = \frac{2n}{2} \cdot n = n^2,$ Q.E.D.

Geometric progression

A geometric progression is a numerical sequence, the first term of which is nonzero, and each next, starting from the second, is equal to the previous one, multiplied by the same number, not equal to zero.

Denoted $\div$ $b_1, b_2, b_3, ..., b_n.$

The attitude of any member $\div$ to its previous term is called the *denominator of the progression* $q = \frac{b_n}{b_{n-1}}.$

Characteristic property of a geometric progression:

Each member $\div\div$, starting from the second is the geometric mean between $b_n = \sqrt{b_{n-1} \cdot b_{n+1}}, n \in Z.$

N-th term formula $\div\div$ $b_n = b_1 \cdot q^{n-1}$, where b_1 – first term, q – denominator, n – member number.

Formulas for the sum of the first *n* terms $\div\div$

$$S_n = \frac{b_n \cdot q - b_1}{q-1}, (q \neq 1). \; S_n = \frac{b_1 \cdot (q^n - 1)}{q-1}.$$

The product of the members $\div\div$, equidistant from its ends, there is a constant value, that is, $b_1 \cdot b_n = b_2 \cdot b_{n-1} = \ldots$

Given $\div\div$ $b_1 = 6,\; q = 3,\; n = 8.$ To find b_n and S_n.

Solution:

According to the formula *n*-th term $\div\div$ $b_n = b_1 \cdot q^{n-1}.$

$$b_8 = 6 \cdot 3^{8-1} = 6 \cdot 3^7 = 6 \cdot 2187 = 13122.$$

$$S_n = \frac{b_1 \cdot (q^n - 1)}{q-1}; \; S_8 = \frac{6 \cdot (3^8 - 1)}{3-1} = \frac{6 \cdot (6561 - 1)}{2} = 19680.$$

Answer: $b_8 = 13122;\; S_8 = 19680.$

Given $\div\div$ $q = 2, n = 7, S_n = 635.$ To find b_1 and b_n.

Solution:

$$b_7 = b_1 \cdot 2^6 = 64 b_1; \; S_n = \frac{b_n q - b_1}{q-1}; \; 65 = \frac{64 \cdot b_1 \cdot 2 - b_1}{2-1}; \; \frac{128 b_1 - b_1}{1} = 635; \; 127 b_1 = 635; \; b_1 = \frac{635}{127};$$

$b_1 = 5.\; b_7 = 64 \cdot 5 = 320.$

Answer: $5; 320.$

Given $\div\div$ $b_1 + b_3 = 15,\; b_2 + b_4 = 30.$ To find S_{10}.

Solution:

Let us solve the following system of equations:

$$\begin{cases} b_1 + b_3 = 15, \\ b_2 + b_4 = 30. \end{cases}$$

Let us express the members $\div\div$, entering the system through b_1 and q:

$$\begin{cases} b_1 + b_1 \cdot q^2 = 15, \\ b_1 \cdot q + b_1 \cdot q^3 = 30. \end{cases} \quad \begin{cases} b_1 \cdot (1 + q^2) = 15, & (1) \\ b_1 \cdot q \cdot (1 + q^2) = 30. & (2) \end{cases}$$

Divide equation (2) by (1):

$$\frac{b_1 \cdot q \cdot (1 + q^2)}{b_1 \cdot (1 + q^2)} = \frac{30}{15}; \; q = 2.$$

$b_1 \cdot (1+2^2) = 15, b_1 = 3.$

According to the formula $S_n = \dfrac{b_1 \cdot (q^n - 1)}{q-1}$, $S_{10} = \dfrac{3 \cdot (2^{10} - 1)}{2-1} = 3 \cdot (1024 - 1) = 3069.$

Answer: 3069.

Find four numbers that form a geometric progression if the first number is greater than the second by 36, and the third is greater than the fourth by 4.

Solution:

According to the statement of the problem $b_1 - b_2 = 36,\ b_1 = b_2 + 36 = b_1 \cdot q + 36;$

$b_3 - b_4 = 4,\ b_3 = b_4 + 4;\ b_1 \cdot q^2 = b_1 \cdot q^3 + 4;\ b_1 \cdot q^2 - b_1 \cdot q^3 = 4;\ b_1 \cdot q \cdot (1-q) = 4.$

Consider such a system of equations: $\begin{cases} b_1(1-q) = 36, & (1) \\ b_1 q^2 (1-q) = 4. & (2) \end{cases}$

Divide (2) by (1):

$\dfrac{b_1 q^2 (1-q)}{b_1(1-q)} = \dfrac{4}{36};\ q^2 = \dfrac{1}{9};\ q_1 = -\dfrac{1}{3};\ q_2 = \dfrac{1}{3}.$

If $q_1 = -\dfrac{1}{3}$, then $b_1 = \dfrac{36}{1 - \left(-\dfrac{1}{3}\right)} = \dfrac{36}{1\frac{1}{3}} = \dfrac{36}{\frac{4}{3}} = \dfrac{36}{1} \cdot \dfrac{3}{4} = 27.$

Then $b_2 = 27 \cdot \left(-\dfrac{1}{3}\right) = -9;\ b_3 = -9 \cdot \left(-\dfrac{1}{3}\right) = 3;\ b_4 = 3 \cdot \left(-\dfrac{1}{3}\right) = -1.$

If $q = \dfrac{1}{3}$, то $b_1 = \dfrac{36}{1 - \frac{1}{3}} = \dfrac{36}{\frac{2}{3}} = 24$, then $b_2 = 54 \cdot \dfrac{1}{3} = 18,\ b_3 = 18 \cdot \dfrac{1}{3} = 6;\ b_4 = 6 \cdot \dfrac{1}{3} = 2.$

Answer: $27; -9; 3; -1;\ \ 54; 18; 6; 2.$

Infinitely decreasing geometric progression

A geometric progression is called infinitely decreasing if its denominator $|q| < 1$.

The sum of its members $S = \dfrac{b_1}{1-q}$, then b_1 – first term :.

Infinitely decreasing geometric progression – I.D.G.P.

Some applications *I.D.G.P.*:

Given : $2; -\dfrac{2}{3}; \dfrac{2}{9}; \ldots$ To find S.

Solution:

Find the denominator : $q = \dfrac{-\frac{3}{2}}{2} = -\dfrac{1}{3}.$

According to the formula $S = \dfrac{b_1}{1-q}$ calculate $S = \dfrac{2}{1 - \left(-\dfrac{1}{3}\right)} = \dfrac{2}{1\frac{1}{3}} = \dfrac{2}{\frac{4}{3}} = \dfrac{3}{2} = 1{,}5.$

Answer: 1,5.

Convert Decimal Periodic Fraction $0{,}46(5)$ to the usual.

Solution:

This fraction can be represented as:

$$0{,}46(5)=0{,}46555...=0{,}46+0{,}005+0{,}0005+0{,}00005+...=\frac{46}{100}+\frac{5}{1000}+\frac{5}{10000}+\frac{5}{100000}+...=$$

$$=\frac{46}{100}+\frac{5}{1000}+\frac{5}{10000}+\frac{5}{100000}+...=\frac{46}{100}+\left(\underbrace{\frac{5}{1000}+\frac{5}{10000}+\frac{5}{100000}+...}_{\text{сума Н.С.Г.П.}}\right)=\frac{5}{10000}:\frac{5}{1000}=\frac{1}{10}.$$

According to the formula $S=\frac{b_1}{1-q}$ we have:

$$S=\frac{\frac{5}{1000}}{1-\frac{1}{10}}=\frac{5}{1000}\cdot\frac{10}{9}=\frac{5}{900};$$

$$=\frac{46}{100}+\frac{5}{900}=\frac{414+5}{900}=\frac{419}{900}.$$

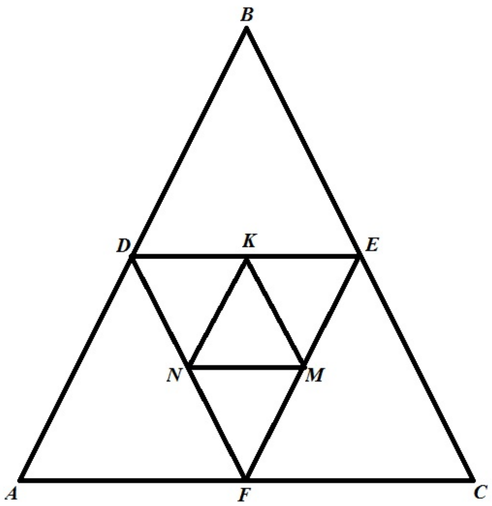
Answer: $\frac{419}{900}$.

Task. In an equilateral triangle with sides b entered a new triangle so that its vertices are the midpoints of the sides of the triangle; a new triangle is inscribed in this triangle in the same way, etc..

Prove that the sequence formed by areas of triangles represents *I.D.G.P.* and Find the sum of these areas.

(I.D.G.P – infinitely decreasing geometric progression)

Solution:



$$S_1=S_{\Delta ABC}=\frac{b^2\sqrt{3}}{4};\quad S_2=S_{\Delta DEF}=\frac{DE^2\sqrt{3}}{4}=\frac{\left(\frac{b}{2}\right)^2\sqrt{3}}{4}=\frac{b^2\sqrt{3}}{16};$$

$$S_3=S_{\Delta KMN}=\frac{MN^2\sqrt{3}}{4}=\frac{\left(\frac{b}{4}\right)^2\sqrt{3}}{4}=\frac{b^2\sqrt{3}}{16\cdot 4}=\frac{b^2\sqrt{3}}{64}\ \text{etc.}$$

$$\frac{S_2}{S_1}=\frac{b^2\sqrt{3}}{16}:\frac{b^2\sqrt{3}}{4}=\frac{b^2\sqrt{3}}{16}\cdot\frac{4}{b^2\sqrt{3}}=\frac{1}{4};\quad \frac{S_3}{S_2}=\frac{b^2\sqrt{3}}{64}:\frac{b^2\sqrt{3}}{16}=\frac{1}{4}.$$

Since a constant value and less than 1, then $S_1, S_2, S_3, \ldots -$ *Н.С.Г.П*

The sum of the areas of the triangles $S = \dfrac{S_1}{1-q} = \dfrac{\frac{b^2\sqrt{3}}{4}}{1-\frac{1}{4}} = \dfrac{\frac{b^2\sqrt{3}}{4}}{\frac{3}{4}} = \dfrac{b^2\sqrt{3}}{3}.$

Answer: $\dfrac{b^2\sqrt{3}}{3}.$

Arithmetic and geometric progressions

Solving problems in which arithmetic and geometric progressions appear at the same time present certain difficulties. Difficulties can be prevented if such a propaedeutic work is carried out:

1) under what condition the numbers $a_1, a_2, \text{i}\, a_3$ form an arithmetic progression?

Answer: three numbers form $\div$ when $a_2 - a_1 = a_3 - a_2$.

2) under what condition the numbers $b_1, b_2, \text{i}\, b_3$ form a geometric progression?

Answer: three numbers form ∺ then when $\dfrac{b_2}{b_1} = \dfrac{b_3}{b_2}$.

3) if the problem is about $\div$ and ∺, then the notation of the unknowns must be introduced only through the terms of one of them.

Task. Three numbers, totaling 21, make up an arithmetic progression.
If we add 2, 3 and 9 to these numbers, respectively, we get a geometric progression. Find these numbers.

Solution

The first condition of the problem can be written as:

$\div\, a_1, a_2, a_3 \quad d -$ its difference.

$\div\, a_1; a_1 + d,\ a_1 + 2d.\ a_1 + a_1 + d + a_1 + 2d = 21,\ 3a_1 + 3d = 21 \,|{:}\, 3\ \ a_1 + d = 7,\ a_1 = 7 - d\ (1)$

If these three members $\div$ respectively add the numbers 2, 3 and 9, we obtain a number of:

$a_1 + 2;\ a_1 + d + 3;\ + a_1 + 2d + 9,$ which form a geometric progression.

Based on characteristic property ∺ we have:

$(a_1 + d + 3)^2 = (a_1 + 2)\cdot(a_1 + 2d + 9).$ Using equality (1), we replace a_1 :

$(7 - d + d + 3)^2 = (7 - d + 2)\cdot(7 - d + 2d + 9);$

$10^2 = (9 - d)\cdot(16 + d);\ 100 = 144 + 9d - 16d - d^2;$

$+ d^2 + 7d - 144 + 100 = 0;\ d^2 + 7d - 44 = 0.$

By Vieta's theorem $d_1 = -11;\ d_2 = 4.$

$a_1 = 7 - d;\ a_1 = 7 - (-11) = 18;\ a_2 = 18 + (-11) = 7;\ a_3 = 7 - 11 = -4.$

$a_1 + a_2 + a_3 = 18 + 7 - 4 = 21.$

Adding to these numbers 2; 3; 9 should receive $18 + 2 = 20;\ 7 + 3 = 10;\ 9 - 4 = 5.$

They are members of a geometric progression because $10^2 = 20 \cdot 5$.
If $d = 4$, then $a_1 = 7 - 4 = 3$; $a_2 = 3 + 4 = 7$; $a_3 = 7 + 4 = 11$; $3 + 7 + 11 = 21$.
$b_1 = 3 + 2 = 5$; $b_2 = 7 + 3 = 10$; $b_3 = 11 + 9 = 20$.
A characteristic property $10^2 = 20 \cdot 5$.
Answer: $18; 7; -4$ or $3; 7; 11$.

The following advice is also useful: if in one progression the given consecutive terms are given, and in the second they are not consecutive, then the unknowns must be denoted through the members of the second progression.
Task. Three numbers whose sum is equal to 114, the successive terms of a geometric progression or 1st, 4th and 25th members arithmetic progression. Find these numbers.

Solution:

Let be a_1 and $d-$ the first term and the difference of the arithmetic progression, then $a_4 = a_1 + 3d$; $a_{25} = a_1 + 24d$.
According to the condition of the problem, we compose the equation:
$a_1 + a_1 + 3d + a_1 + 24d = 114$; $3a_1 + 27d = 114 \,|\,:3$ $a_1 + 9d = 38 \to a_1 = 38 - 9d$ (1)
On the other hand, these same numbers are consecutive members of a geometric progression, then
$(a_1 + 3d)^2 = a_1 \cdot (a_1 + 9d)$. Using equality (1), we have:
$(38 - 9d + 3d)^2 = (38 - 9d) \cdot (38 - 9d + 24d)$; $(38 - 6d)^2 = (38 - 9d) \cdot (38 + 15d)$;
$1444 - 456d + 36d^2 = 1444 + 570d - 342d - 135d^2$; $-456d + 36d^2 - 570d + 32d + 135d^2 = 0$;
$171d^2 - 684d = 0 \,|\,:171$ $d^2 - 4d = 0$; $d(d - 4) = 0$;
$d = 0 -$ does not satisfy the condition of the problem.
$d = 4$. Если $d = 4$, то $a_1 = 38 - 9 \cdot 4 = 2$ *(see equation (1))*
$a_4 = 2 + 3 \cdot 4 = 14$; $a_{25} = 2 + 24 \cdot 4 = 98$. $2 + 14 + 98 = 114$.
Answer: $2; 14; 98$.
Task. The four numbers make up an arithmetic progression. If we subtract from them the number 2; 6; 7; 2, then we get numbers forming a geometric progression. Find these numbers.

Solution:

Let's denote the unknown numbers as members of the geometric progression:
$b_1 -$ first term, $q -$ denominator $\div$ then $b_1 \cdot q -$ second term, $b_1 \cdot q^2 -$ third term, $b_1 \cdot q^3 -$ fourth term. Adding to them, respectively, the numbers 2; 6; 7; 2, we get the members of the arithmetic progression:
$\div b_1 + 2$; $b_1 q + 6$; $b_1 q^2 + 7$; $b_1 q^3 + 2$.
Using twice the characteristic property of the arithmetic progression, we obtain the following system of equations:

$$\begin{cases} b_1q+6=\dfrac{b_1q+7+b_1+2}{2}, & \Big|\cdot 2 \\ b_1q^2+7=\dfrac{b_1q^3+2+b_1q+6}{2} & \Big|\cdot 2 \end{cases} \rightarrow \begin{cases} 2\cdot(b_1q+6)=b_1q^2+7+b_1+2, \\ 2\cdot(b_1q^2+7)=b_1q^3+2+b_1q+6. \end{cases}$$

$$\begin{cases} 2\cdot b_1q+12-b_1q^2-9-b_1=0, \\ 2\cdot b_1q+14-b_1q^3-8-b_1q=0. \end{cases} \rightarrow \begin{cases} 2\cdot b_1q-b_1q^2-b_1+3=0, \\ 2\cdot b_1q^2-b_1q^3-b_1q+6=0. \end{cases}$$

$$\begin{cases} 2\cdot b_1q-b_1q^2-b_1=-3, \\ 2\cdot b_1q^2-b_1q^3-b_1q=-6. \end{cases} \rightarrow \begin{cases} 2\cdot b_1q-b_1q^2-b_1=-3, & (1) \\ q\cdot(2b_1q-b_1q^2-b_1)=-6. & (2) \end{cases}$$

Divide (2) by (1):

$$\frac{q\cdot(2b_1q-b_1q^2-b_1)}{2b_1q-b_1q^2-b_1}=\frac{-6}{-3},\ q=2.$$

Substitute the value q into the equation (1):

$$2\cdot b_1\cdot 2-b_1\cdot 2^2-b_1=-3;\ 4b_1-4b_1-b_1=-3,\ b_1=3.$$

$$b_2=3\cdot 2=6,\ b_3=3\cdot 2^2=12,\ b_4=3\cdot 2^3=24.$$

Answer: $3; 6; 12; 24.$

Task. Find the arithmetic and geometric progression if it is known that the first, third and eleventh members of the arithmetic progression are twice the first, third and fifth members of the geometric progression, and the sum of the first members of the arithmetic progression is 40.

Solution:

Let the a_1 and the first term and the difference ÷,

b_1 and $q-$ first term and denominator ∺.

According to the statement of the problem $a_1=2\cdot b_1,\ a_2=2\cdot b_3,\ a_{11}=2\cdot b_5,$

$$S_5=\frac{a_1+a_5}{2}\cdot 5=40.$$

You can create such a system of equations:

$$\begin{cases} a_1=2b_1, \\ a_1+2d=2b_1\cdot q^2, \\ a_1+10d=2b_1\cdot q^4, \\ \dfrac{a_1+a_5}{2}=\dfrac{40}{5}. \end{cases}$$

After all $a_3=a_1+2d,\ b_3=b_1\cdot q^2,\ a_{11}=a_1+10d,\ b_5=b_1\cdot q^4.$

Because a_1 and a_5- terms equidistant from the ends of the arithmetic progression, then $\dfrac{a_1+a_5}{2}=\dfrac{a_2+a_4}{2}=8=a_3=a_1+2d.$

$$\begin{cases} a_1=2b_1, & (1) \\ a_1+2d=2b_1\cdot q^2, & (2) \\ a_1+10d=2b_1\cdot q^4, & (3) \\ a_1+2d=8. & (4) \end{cases}$$

Substituting (1) into (2), (3), (4), we obtain a system of three equations:

$$\begin{cases} 2b_1+2d=2b_1\cdot q^2, & |:2 \\ 2b_1+10d=2b_1\cdot q^4, & |:2 \\ 2b_1+2d=8. & |:2 \end{cases} \quad \begin{cases} b_1+d=b_1q^2, & (5) \\ b_1+5d=b_1q^4, & (6) \\ b_1+d=4. & (7) \end{cases}$$

From equation (7) we have $d=4-b_1$ (8).

Substituting (7) into (5) and (6) taking into account (8), we obtain:

$$\begin{cases} 4=b_1\cdot q^2, \\ 4+4\cdot(4-b_1)=b_1\cdot q^4 \end{cases} \quad \text{because } b_1+5d=b_1+d+4d.$$

$$\begin{cases} 4=b_1\cdot q^2, \\ 4+16-4b_1=b_1\cdot q^4 \end{cases} \quad \begin{cases} 4=b_1\cdot q, & (9) \\ 20-4b_1=b_1\cdot q^4 & (10) \end{cases}$$

Substituting (9) into (10):

$20-4b_1=b_1\cdot q^2\cdot q^2=4q^2.$ From equation (9) we have:

$\cdot q^2=\frac{4}{b_1}.$ Then $20-4b_1=4\cdot\frac{4}{b_1};\ 20-4b_1=\frac{16}{b_1};\ b_1\neq 0.$

$20b_1-4b_1^2=16;\ 20b_1-4b_1^2-16=0\,|:(-4)\ \ b^2-5b_1+4=0;$

By Vieta's theorem $b_1=1;\ b_2=4.$

If $b=1,$ then based on the equation (8) $d=4-1=3$ и $a_1=2b_1=2\cdot1=2;\ a_2=2+3=5;$ $a_3=5+3=8.\ \div$

If $b=4,$ then $a_1=2\cdot4=8;\ a_2=8+3=11;\ a_3=11+3=14.\ \div$

Find the denominator q from equation (5):

If $b_1=1,$ then $1+3=1\cdot q^2;\ q_1=-2;\ q_2=2.$

$b_2=1\cdot(-2)=-2;\ b_3=(-2)\cdot(-2)=4;\ b_4=4\cdot(-2)=-8.$

$b_2=1\cdot2=2;\ b_3=2\cdot2=4;\ b_4=4\cdot2=8.$

If $b_1=4,$ then $4+3=4\cdot q^2;\ q^2=\frac{7}{4};\ q_1=-\frac{\sqrt7}{2};\ q_2=\frac{\sqrt7}{2}.$

$$b_2=4\cdot\left(-\frac{\sqrt7}{2}\right)=-2\sqrt7;\ b_3=-2\sqrt7\cdot\left(-\frac{\sqrt7}{2}\right)=7;\ b_4=7\cdot\left(-\frac{\sqrt7}{2}\right)=-\frac{7\sqrt7}{2}.$$

$$b_2=4\cdot\frac{\sqrt7}{2}=2\sqrt7;\ b_3=2\sqrt7\cdot\frac{\sqrt7}{2}=7;\ b_4=7\cdot\frac{\sqrt7}{2}=\frac{7\sqrt7}{2}.$$

Answer: $2;5;8;\ -2;4;8;\ 4;-2\sqrt7;7;-\frac{7\sqrt7}{2};\ 4;2\sqrt7;7;\frac{7\sqrt7}{2}.$

Find four numbers, of which the first three form an arithmetic progression, and the last three form a geometric progression. The sum of the two extreme numbers is 37, and the sum of the two middle numbers is 36.

Solution:

We denote $x-$ first number, $y-$ second number. Then $36-y-$ third number.

$x;y;36-y-$ arithmetic progression.

By its characteristic property, we have:

$2y=x+36-y\,(1)$

According to the condition of the problem, we compose the second equation:

$y; 36-y; 37-x-$ geometric progression.

$(36-y)^2 = y\cdot(37-x)(2)$

From equations (1) and (2) we form the system:

$\begin{cases} 2y = x+36-y, \\ (36-y)^2 = y\cdot(37-x) \end{cases}$ $\begin{cases} 3y = x+36, \\ (36-y)^2 = y\cdot(37-x) \end{cases}$ $\begin{cases} x = 3y-36, \\ (36-y)^2 = y\cdot(37-3y+36) \end{cases}$

$\begin{cases} x = 3y-36, \\ 1296-72y+y^2 = 73y-3y^2 \end{cases}$

$4y^2-145y+1296=0;\ D=21025-20736=289=17^2.$

$y_1 = \frac{145-17}{16} = \frac{128}{16} = 8.\ y_2 = \frac{145+17}{16} = \frac{162}{16} = 20{,}25.$

$x_1 = 3\cdot16-36=12;\ x_2 = 3\cdot20{,}25 = 60{,}75-36 = 24{,}75.$

$12; 16; 36-16; 37-12;\ \ 24{,}75; 20{,}25; 36-20{,}25; 15{,}75; 12{,}25; 37-24{,}75;$

$12; 16; 20; 25;\ \ 24{,}75; 20{,}25; 15{,}75; 12{,}25.$

Answer: $12; 16; 20; 25$ або $24{,}75; 20{,}25; 15{,}75; 12{,}25.$

Task. Find the sum of the first terms of an arithmetic progression if its $(m+1)$ term is $2m+1$.

Solution:

Find the first and second term of the progression:

$a_{m+1} = 2m+1.$

$a_1 = a_{0+1} = 2\cdot0+1=1;\ a_2 = a_{1+1} = 2\cdot1+1=3,$ then $d = a_2-a_1 = 3-1 = 2.$

$S_n = \frac{2a_1+d\cdot(n-1)}{2}\cdot n;\ S_n = \frac{2\cdot1+2\cdot(n-1)}{2}\cdot n = \frac{2+2n-2}{2} = n^2.$

Answer: n^2.

Prove that if $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{b+a}$ form an arithmetic progression, then the numbers a^2, b^2, c^2 also form an arithmetic progression.

Solution:

By the property of the members of the arithmetic progression:

$\frac{1}{c+a} - \frac{1}{b+c} = \frac{1}{b+a} - \frac{1}{c+a}$. Reduce the fractions on the left and right sides of the equation to common denominators:

$\frac{b+c-a-c}{(c+a)\cdot(b+c)} = \frac{c+a-b-a}{(b+a)\cdot(c+a)};\ \frac{b-a}{(c+a)\cdot(b+c)} = \frac{c-b}{(c+a)\cdot(b+c)};$

$\frac{b-a}{(c+a)\cdot(b+c)} = \frac{c-b}{(b+a)\cdot(c+a)}\ |\cdot(b+a)\cdot(c+a)$

$(b-a)\cdot(b+a) = (c-b)\cdot(c+b);\ b^2-a^2 = c^2-b^2;\ 2b^2 = c^2+a^2;\ b^2 = \frac{c^2+a^2}{2}-$ this is a characteristic property of the arithmetic progression, that is, the numbers a^2, b^2, c^2 form an arithmetic progression.

Self-study assignments:

Given ÷ The values of three quantities are known. Find the other two.

№	a_1	d	n	a_n	S_n	Answers:	
1	10	4	11			$a_n=50$	$S_n=330$
2	10	4		50		$n=11$	$S_n=330$
3	10	4			330	$n=11$	$a_n=50$
4	10		11		330	$d=4$	$a_n=50$
5		4		50	330	$a_1=10;-6$	$n=11;15$
6	110	-10	11			$a_n=10$	$S_n=660$
7	4	$-\frac{1}{4}$	13			$a_n=1$	$S_n=32{,}5$
8	5		26	105		$d=4$	$S_n=1430$
9	$\frac{3}{4}$		26	$3\frac{7}{18}$		$d=\frac{1}{20}$	$S_n=56\frac{7}{9}$
10		3	12		210	$a_1=1$	$a_n=34$
11		2	15	-10		$a_1=-38$	$S_n=-360$
12	0	0,5		5		$n=11$	$S_n=27{,}5$
13	-9	$\frac{1}{2}$			-75	$n=25$	$a_n=3$
14	-28		9		0	$d=7$	$a_n=28$
15	0,2			5,2	137,7	$d=0{,}1$	$n=51$
16			30	$15\frac{3}{4}$	$146\frac{1}{4}$	$a_1=-6$	$d=\frac{3}{4}$
17		0,3		50,3	2551,3	$a_1=32$	$n=62$

÷ $a_1+a_2+a_3=9;\ a_1+a_2+a_3+a_4=16,\ S_n=100.\ n-?$ Answer: 10.

÷ $a_3=9;\ a_7-a_2=20;\ S_n=91.\ n-?$ Answer: 7.

÷ increasing, $a_1+a_2+a_3=27;\ a_1^2+a_2^2+a_3^2=275;\ a_1=?\ d=?$
Answer: $a_1=5, d=4$.

÷ $a_3+a_5=14;\ S_{12}=129;\ S_n=195.\ n=?$ Answer: 15.

÷ increasing, $a_1+a_7=4;\ a_3^2+a_7^2=122;\ a_1=?\ d=?$ Answer: $a_1=-7, d=3$.

÷ increasing, $a_5:a_3=4;\ a_2\cdot a_6=(-11);\ a_1=?\ d=?$ Answer: $a_1=-4, d=3$.

Find the first term and the difference ÷:

$\begin{cases} a_2 + a_8 = 10, \\ a_3 + a_{14} = 31. \end{cases}$ Answer: $a_1 = -7; d = 3.$

$\begin{cases} S_5 - S_2 - a_5 = 0{,}1, \\ S_4 + a_7 = 0{,}1. \end{cases}$ Answer: $a_1 = 1{,}05; d = 0{,}8.$

$\begin{cases} S_4 = 9, \\ S_6 = 22\frac{1}{2}. \end{cases}$ Answer: $a_1 = 0{,}25; d = 1{,}5.$

$\begin{cases} a_3 + a_5 + a_8 = 18, \\ a_4 + a_2 = -2. \end{cases}$ Answer: $a_1 = 1\frac{2}{3}; d = -1\frac{1}{3}.$

$\div\div\ 10, 20, 40, \ldots\ S_{10} - ?$ Answer: $10230.$

$-4, 16, -64, \ldots\ S_7 - ?$ Answer: $-13108.$

$3, -1, \frac{1}{3}, \ldots\ S_8 - ?$ Answer: $2\frac{182}{729}.$

$b_1 = 5,\ q = -\frac{1}{5},\ n = 6.$ To find b_n and S_n. Answer: $-\frac{1}{625};\ 4\frac{104}{625}.$

$b_n = 128,\ q = 2,\ n = 7.$ To find b_1 and S_n. Answer: $2;\ 254.$

$b_1 = 3,\ q = 2,\ b_n = 96.$ To find n and S_n. Answer: $6;\ 189.$

$b_1 = 81,\ b_n = -10\frac{2}{3},\ n = 6.$ To find q and S_n. Answer: $-\frac{2}{3};\ 44\frac{1}{3}.$

Insert three such numbers between the numbers 1 and 16 so that they, together with these numbers, form a geometric progression.
Answer: 1; 2; 4; 8; 16.
I.D.G.P.: $1; \frac{1}{2}; \frac{1}{4}; \ldots$ Find the amount. Answer: 2.

$1 - \frac{1}{3} + \frac{1}{9} - \ldots\ S - ?.$ Answer: $0{,}75.$

$1 - \frac{1}{2} + \frac{1}{4} - \ldots\ S - ?.$ Answer: $\frac{2}{3}.$

A square is inscribed in a circle with radius a. A circle is inscribed in the square; in a circle - a square, etc..
Find the sum of the areas of all circles and the sum of the areas of all squares.

Answer: $2\pi a^2$; $4a^2$.

Task. Write the first three terms of an arithmetic progression in which the sum of an arbitrary number of terms is given by the formula $S_n = 7n^2 - 5n.$

Answer: 2; 16; 30.

Task. The sum of the three numbers that form an arithmetic progression is 15. If we add to them the numbers 1, 4 and 19, respectively, we get three numbers that form a geometric progression. Find these numbers.

Answer: 2; 5; 8 і 26; 5; -16.

Task. The sum of three numbers, which are consecutive members of the arithmetic progression, is equal to 21. If the second number is reduced by one, and the third is increased by one, then three consecutive members of the geometric progression are formed. Find these numbers.

Answer: 3; 7; 11 и 12; 7; 2.

At what values α numbers $2 \cdot \cos\frac{\pi}{6}$, $4 \cdot \sin\alpha$, $6 \cdot \sin(\pi - \alpha)$ are consecutive members of an arithmetic progression?

Answer: $\alpha = (-1)^k \cdot \frac{\pi}{3} + k\pi, k \in Z.$