Section 18
Systems of trigonometric equations

Systems of trigonometric equations can be divided into two large types:

a) those in which it is relatively easy to eliminate one of the unknowns by expressing it in terms of other variables from any equation of the system;

б) those in which the trigonometric system can be reduced to a system of algebraic equations in which trigonometric functions are taken as new variables.

When solving systems of trigonometric equations, as well as when solving trigonometric equations with one variable, one should try to carry out such identical transformations in which one or more equations break down into simple equations with two variables, for example, $\cos(2x+3y)=1,\ ctg(x-y)=\sqrt{3}$ etc.
Let us illustrate the above with examples..
Solve system of equations:

$$\begin{cases}\sqrt{\sin x}\cdot\cos y=0, & (1)\\ 2\sin^2 x-\cos 2y-2=0. & (2)\end{cases}$$

Solution:

Equation (1) implies that $\sin x\ge 0.$
If $\sin x=0,\ 0\cdot\cos y=0-$ identity.
If $\sin x>0,$ then it follows from equation (1) that $\cos y=0.$
Thus, the original system turns into a combination of two systems:

$$\left[\begin{array}{l}\begin{cases}\sin x=0,\\ 2\sin^2 x-\cos 2y-2=0.\end{cases}\\ \begin{cases}\cos y=0,\ \sin x>0,\\ 2\sin^2 x-\cos 2y-2=0.\end{cases}\end{array}\right.\quad \left[\begin{array}{l}\begin{cases}\sin x=0,\\ 0-\cos 2y-2=0.\end{cases}\\ \begin{cases}\cos y=0,\\ 2\sin^2 x-\cos 2y-2=0,\\ \sin x>0.\end{cases}\end{array}\right.\quad \left[\begin{array}{l}\begin{cases}\sin x=0,\\ -\cos 2y=2.\end{cases}\quad\varnothing\\ \begin{cases}\cos y=0,\\ 2\sin^2 x-2=0,\\ \sin x>0.\end{cases}\end{array}\right.$$

$$\begin{cases}\cos y=0,\\ 2\sin^2 x=2.\end{cases}\quad \begin{cases}\cos y=0,\\ 2\cdot\dfrac{1-\cos 2x}{2}=2.\end{cases}\quad \begin{cases}\cos y=0,\\ 1-\cos 2x=2.\end{cases}\quad \begin{cases}\cos y=0,\\ \cos 2x=-1.\end{cases}\quad \begin{cases}y=\dfrac{\pi}{2}+\pi n,\ n\in Z.\\ x=\dfrac{\pi}{2}+\pi k,\ n\in Z.\end{cases}$$

Answer: $\left(\dfrac{\pi}{2}+\pi k;\ \dfrac{\pi}{2}+\pi n\right),\ k,n\in Z.$

$$\frac{3+2\cdot\cos(x-y)}{2}=\sqrt{3+2x-x^2}\cdot\cos^2\cdot\frac{x-y}{2}+\frac{\sin^2(x-y)}{2}.$$

Solution:

Lower the degree of the cosine:

$$\frac{3+2\cdot\cos^2(x-y)}{2}=\sqrt{3+2x-x^2}\cdot\frac{1+\cos\left(2\cdot\dfrac{x-y}{2}\right)}{2}+\frac{\sin^2(x-y)}{2};$$

$$\frac{3+2\cdot\cos^2(x-y)}{2}=\sqrt{3+2x-x^2}\cdot\frac{1+\cos(x-y)}{2}+\frac{\sin^2(x-y)}{2};\Big|\cdot 2$$

$3+2\cdot\cos^2(x-y)=\sqrt{3+2x-x^2}\cdot(1+\cos(x-y))+\sin^2(x-y);$

$3+2\cdot\cos^2(x-y)-\sqrt{3+2x-x^2}\cdot(1+\cos(x-y))-\sin^2(x-y)=0;$

$1-\sin^2(x-y)+2+2\cos(x-y)-\sqrt{3+2x-x^2}+\cos(x-y)\cdot\sqrt{3+2x-x^2}=0;$

$\cos^2(x-y)+2-\sqrt{3+2x-x^2}+\cos(x-y)\cdot\left(2-\sqrt{3+2x-x^2}\right)=0;$

$\cos^2(x-y)+\left(2-\sqrt{3+2x-x^2}\right)\cdot(1-\cos(x-y))=0.$

Let's introduce new variables:

$\cos(x-y)=t,\ 2-\sqrt{3+2x-x^2}=a,$ then $t^2+a\cdot(1-t)=0,\ t^2+a-at=0,\ t^2-at+a=0-$ this is a quadratic equation with respect to t it has roots when $D=a^2-4a\ge 0,\ a(a-4)\ge 0.$

This inequality is possible when $\begin{cases}a>0,\\ a-4\ge 0.\end{cases}$ or $\begin{cases}a\le 0,\\ a-4\le 0.\end{cases}$

$\begin{cases}a\ge 0,\\ a\ge 4.\end{cases}\to a\ge 4 \qquad \begin{cases}a\le 0,\\ a\le 4.\end{cases}\to a\le 0.$

Hence, the equation $t^2-at+a=0$ has roots at $a\in(-\infty;0]\cup[4;+\infty).$

In that $a=2-\sqrt{3+2x-x^2}=2-\sqrt{-(x^2-2x-3)}=2-\sqrt{-((x^2-2x+1)+1-3)}=$
$=2-\sqrt{-((x-1)^2-4)}=2-\sqrt{4-(x-1)^2}.$

$2-2-\sqrt{4}\le a\le 2.\ a=0.$

$2-\sqrt{4-(x-1)^2}=0.\ 2^2=\sqrt{4-(x-1)^2};\ 4=4-(x-1)^2;\ (x-1)^2=0,\ x-1=0,\ x=1.$

Then the equation: $\cos^2(x-y)+\left(2-\sqrt{3+2x-x^2}\right)\cdot(1-\cos(x-y))=0$ takes the form:

$\cos^2(1-y)=0,\ \cos^2(y-1)=0;\ y-1=\frac{\pi}{2}+\pi n,\ n\in Z.\ y=1+\frac{\pi}{2}+\pi n,\ n\in Z.$

Answer: $\left(1;\ 1+\frac{\pi}{2}+\pi n\right),\ n\in Z.$

$\begin{cases}ctgx+\sin 2y=\sin 2x,\\ 2\sin y\cdot\sin(x+y)=\cos x.\end{cases}$

Solution:

We transform the second equation of the system by applying the transformation formula for the product of trigonometric functions:

$2\sin y\cdot\sin(x+y)=2\cdot\frac{\cos(y-x-y)-\cos(y+x+y)}{2}=\cos(-x)-\cos(x+2y)=\cos x-\cos(x+2y).$

$\cos x-\cos(x+2y)=\cos x,\ \cos x-\cos x=\cos(x+2y);\ \cos(x+2y)=0,\ x+2y=\frac{\pi}{2}+\pi k;$

$x=\frac{\pi}{2}-2y+\pi k,\ k\in Z.$

Substitute the x values into the first equation of the system:

$ctg\left(\frac{\pi}{2}-2y\right)+\sin 2y=\sin 2\left(\frac{\pi}{2}-2y\right);$

$tg2y+\sin 2y=\sin(\pi-4y);$

$tg2y+\sin 2y=\sin 4y;$

$\frac{\sin 2y}{\cos 2y}+\sin 2y-\sin 4y=0\,|\cdot\cos 2y;$

$\sin 2y + \sin 2y - 2\sin 2y \cdot \cos 2y \cdot \cos 2y = 0;$
$\sin 2y \cdot \left(1 + \cos 2y - 2\cos^2 2y\right) = 0.$
$\sin 2y \cdot \left(1 + \cos 2y - \cos 2y - \cos 2y\right) = 0;$

$\sin 2y\left(1 - \cos 2y\right) = 0 \quad \left[\begin{array}{l}\sin 2y = 0,\\ 1 - \cos 2y = 0.\end{array}\right. \quad \left[\begin{array}{l}2y = \pi n,\ n \in Z,\\ \cos 2y = 1.\end{array}\right. \quad \left[\begin{array}{l} y = -\dfrac{\pi n}{2} + \pi n,\ n \in Z,\\ 2y = 2\pi n,\ y = \dfrac{\pi n}{2} + n.\end{array}\right.$

Answer: $\left(\dfrac{\pi}{2} + \pi n;\ -\dfrac{\pi}{2} n\right); \left(\dfrac{\pi}{2} + \pi n;\ -\dfrac{\pi}{2} + n\right).$

$$\begin{cases} tgx - tgy = -2\sqrt{3},\\ x - y = \dfrac{\pi}{3}.\end{cases}$$

Solution

We transform the first equation of the system:

$\dfrac{\sin x}{\cos x} - \dfrac{\sin y}{\cos y} = -2\sqrt{3};\ \dfrac{\sin x \cdot \cos y - \cos x \cdot \sin y}{\cos x \cdot \cos y} = -2\sqrt{3};\ \dfrac{\sin(x - y)}{\cos x \cdot \cos y} = -2\sqrt{3};$

$\sin(x - y) = -2\sqrt{3} \cdot \cos x \cdot \cos y;\ \sin(x - y) = -2\sqrt{3} \cdot \dfrac{\cos(x + y) + \cos(x - y)}{2};$

$\sin(x - y) = -\sqrt{3} \cdot \left(\cos(x + y) + \cos(x - y)\right).$

Let us substitute from the second equation of the system the values: $x - y = \dfrac{\pi}{3}$:

$\sin\dfrac{\pi}{3} = -\sqrt{3}\left(\cos(x + y) + \cos\dfrac{\pi}{3}\right);\ \dfrac{\sqrt{3}}{2} = -\sqrt{3}\left(\cos(x + y) + \dfrac{1}{2}\right); \Big|: \left(-\sqrt{3}\right)$

$\cos(x + y) + \dfrac{1}{2} = -\dfrac{1}{2};\ \cos(x + y) = -1;\ x + y = \pi + 2\pi n,\ n \in Z.$

Consider the system of equations:

$$-\begin{cases} x + y = \pi + 2\pi k,\\ x - y = \dfrac{\pi}{3}\end{cases}$$

$2y = \dfrac{2\pi}{3} + 2k\pi \Big|: 2 \quad y = \dfrac{\pi}{3} + k\pi, k \in Z.$

$$+\begin{cases} x + y = \pi + 2\pi k,\\ x - y = \dfrac{\pi}{3}\end{cases}$$

$2x = \dfrac{4\pi}{3} + 2k\pi,\ x = \dfrac{2\pi}{3} + k\pi.$

Answer: $\left(\dfrac{2\pi}{3} + k\pi;\ \dfrac{\pi}{3} + k\pi\right),\ k \in Z.$

$$\begin{cases} \sin\dfrac{x + y}{2} \cdot \sin\dfrac{x - y}{2} = \dfrac{1}{\sqrt{2}},\\ \cos x \cdot \cos y = -\dfrac{1}{2}.\end{cases}$$

Solution:

Converting the product of sines into a sum, we get the equation:

$$\frac{1}{2}\cdot\left(\cos\left(\frac{x+y}{2}-\frac{x-y}{2}\right)-\cos\left(\frac{x+y}{2}+\frac{x-y}{2}\right)\right)=\frac{\sqrt{2}}{2};\ \cos y-\cos x=\sqrt{2}.$$

The original system is equivalent to such a system:

$$\begin{cases}\cos y-\cos x=\sqrt{2},\\ \cos x\cdot\cos y=-\dfrac{1}{2}.\end{cases}$$ Let the $\cos x=a, \cos y=b.$

Then $\begin{cases}b-a=\sqrt{2},\\ a\cdot b=-\dfrac{1}{2}.\end{cases}$ $\begin{cases}b=a+\sqrt{2},\\ a\cdot\left(a+\sqrt{2}\right)=-\dfrac{1}{2}.\end{cases}$ $\begin{cases}b=a+\sqrt{2},\\ a^2+a\sqrt{2}+\dfrac{1}{2}=0.\end{cases}$

$$D=2-4\cdot\frac{1}{2}=0;\ \left(a+\sqrt{2}\right)^2=0;\ a=-\frac{\sqrt{2}}{2};$$

$$b=a+\sqrt{2}=-\frac{\sqrt{2}}{2}+\sqrt{2}=\frac{-\sqrt{2}+2\sqrt{2}}{2}=\frac{\sqrt{2}}{2};\ \cos x=-\frac{\sqrt{2}}{2};\ \cos y=\frac{\sqrt{2}}{2}.$$

$$x=\pm\frac{3\pi}{2}+2\pi m,\ m\in Z;\ y=\pm\frac{\pi}{4}+2\pi n,\ n\in Z.$$

Answer: $\left(\pm\dfrac{3\pi}{2}+2\pi m;\ \pm\dfrac{\pi}{4}+2\pi n\right),\ m\in Z,\ n\in Z.$

$$\begin{cases}\cos\pi x\cdot\cos\pi y=\dfrac{1}{4},\\ tg\pi x\cdot tg\pi y=-3.\end{cases}$$

Solution:

$$\begin{cases}\cos\pi x\cdot\cos\pi y=\dfrac{1}{4},\\ \dfrac{\sin\pi x}{\cos\pi x}\cdot\dfrac{\sin\pi y}{\cos\pi y}=-3.\end{cases}$$ Multiply:

$$\frac{\sin\pi x\cdot\sin\pi y}{\cos\pi x\cdot\cos\pi y}\cdot\cos\pi x\cdot\cos\pi y=\frac{1}{4}\cdot(-3);$$

$$\sin\pi x\cdot\sin\pi y=-\frac{3}{4}.$$

Consider a system equivalent to the original:

$$-\begin{cases}\cos\pi x\cdot\cos\pi y=\dfrac{1}{4},\\ \sin\pi x\cdot\sin\pi y=-\dfrac{3}{4}.\end{cases}$$ Subtract and add:

$$\begin{cases}\cos\pi x\cdot\cos\pi y-\sin\pi x\cdot\sin\pi y=\dfrac{1}{4}+\dfrac{3}{4},\\ \cos\pi x\cdot\cos\pi y+\sin\pi x\cdot\sin\pi y=\dfrac{1}{4}-\dfrac{3}{4}.\end{cases}$$

$$\begin{cases}\cos(\pi x+\pi y)=1,\\ \cos(\pi x-\pi y)=-\dfrac{1}{2}.\end{cases}\ \begin{cases}\pi x+\pi y=2k\pi,\\ \pi x-\pi y=\pm\dfrac{2\pi}{3}+2\pi n.\end{cases}\begin{matrix}|:\pi\\ |:\pi\end{matrix}\ \begin{cases}x+y=2k,\\ x-y=2\pi>\dfrac{2}{3}.\end{cases}\ k\in Z,\ n\in Z.$$

Add and subtract:

$$2x = 2k + 2\pi + \frac{2}{3}\Big|:2 \quad x = k + n + \frac{1}{3},\ y = k - n - \frac{1}{3};\ \ x = k + n - \frac{1}{3};\ \ x = k - n + \frac{1}{3}.$$

Answer: $\left(k + n + \frac{1}{3}; k - n - \frac{1}{3}\right)$, $\left(k + n - \frac{1}{3}; k - n + \frac{1}{3}\right)$.

$$\begin{cases} \sin x - \sin y = \frac{1}{2}, \\ \cos x - \cos y = -\frac{\sqrt{3}}{2}. \end{cases}$$

Solution:

Let us transform the differences of the functions of the same name into products:

$$\begin{cases} 2\cos\frac{x+y}{2}\cdot\sin\frac{x-y}{2} = \frac{1}{2}, & (1) \\ -2\sin\frac{x+y}{2}\cdot\sin\frac{x-y}{2} = -\frac{\sqrt{3}}{2}. & (2) \end{cases}$$

Dividing the second equation by the first, we get:

$$tg\frac{x+y}{2} = \sqrt{3},\ \ \frac{x+y}{2} = \frac{\pi}{3} + k\pi,\ \ k \in Z.$$

Consider two cases:

а) $k = 2n,$ even number, then $\frac{x+y}{2} = \frac{\pi}{3} + 2\pi n, n \in Z.$

Substitute the value $\frac{x+y}{2} = \frac{\pi}{3} + 2\pi n$ into equation (1):

$$2\cos\frac{\pi}{3}\cdot\sin\frac{x-y}{2} = \frac{1}{2},\ \ 2\cdot\frac{1}{2}\cdot\sin\frac{x-y}{2} = \frac{1}{2},$$

$$\sin\frac{x-y}{2} = \frac{1}{2},\ \frac{x-y}{2} = (-1)^n\arcsin\frac{1}{2} + \pi n = (-1)^n\cdot\frac{\pi}{6} + \pi n, n \in Z.$$

Then there is such a system of equations:

$$\begin{cases} \frac{x+y}{2} = \frac{\pi}{3} + k\pi, \\ \frac{x-y}{2} = (-1)^n\cdot\frac{\pi}{6} + \pi n. \end{cases} \quad \begin{cases} \frac{x+y}{2} = \frac{\pi}{3} + k\pi, \\ \frac{x-y}{2} = \left(-\frac{5\pi}{6}\right) + \pi n. \end{cases} \quad \begin{cases} \frac{x+y}{2} = \frac{\pi}{3} + k\pi, \\ \frac{x-y}{2} = \frac{5\pi}{3} + \pi n. \end{cases} \begin{vmatrix} \cdot 2 \\ \cdot 2 \end{vmatrix}$$

$$+\begin{cases} x + y = \frac{2\pi}{3} + 2k\pi, \\ x - y = \frac{10\pi}{3} + 2\pi n. \end{cases} \qquad -\begin{cases} x + y = \frac{2\pi}{3} + 2k\pi, \\ x - y = \frac{10\pi}{3} + 2\pi n. \end{cases}$$

$$2x = \frac{12\pi}{3} + 2k\pi + 2\pi n, \Big|:2 \qquad 2y = -\frac{8\pi}{3} + 2k\pi - 2\pi n,$$

$$x = 2\pi + k\pi + \pi n, \qquad y = -\frac{4}{3}\pi + \pi(k - n),$$

$$x = 2\pi + \pi(k + n), \qquad y = \pi\left(\frac{4}{3} + k - n\right).$$

$$x = \pi(2 + k + n).$$

б) $k = 2n + 1,$ $k -$ odd number.

$\frac{x+y}{2}=2\pi n+\frac{4\pi}{3}.\qquad x=\frac{7\pi}{6}+2\pi(n+k);$

$y=\frac{3\pi}{2}+2\pi(n-k).$

Answer: $\left(\pi(2+k+n);\ \pi\left(-\frac{4}{3}+k-n\right)\right),\ \left(\frac{7\pi}{6}+2\pi(n+k);\ \frac{3\pi}{2}+2\pi(n-k)\right).$

$$\begin{cases}\cos x:\cos y:\cos z=5:4:3,\\ x+y+z=\frac{\pi}{2}.\end{cases}$$

Solution:

Let the $k-$ proportionality coefficient, then $\cos x=5k,\ \cos y=4k,\ coz=3k,\ k\neq 0.$ *(A)*

From the equation $x+y+z=\frac{\pi}{2}$ we have $x+y=\frac{\pi}{2}-z.$

$\sin(x+y)=\sin\left(\frac{\pi}{2}-z\right)=\cos z.$

$y+z=\frac{\pi}{2}-x,\ \sin(y+z)=\sin\left(\frac{\pi}{2}-x\right)=\cos x.$

$x+z=\frac{\pi}{2}-y,\ \sin(z+x)=\sin\left(\frac{\pi}{2}-y\right)=\cos y.$

There is such a system of equations:

$$\begin{cases}\sin(x+y)=\cos z,\\ \sin(y+z)=\cos x,\\ \sin(z+x)=\cos y.\end{cases}\quad\begin{cases}\sin x\cdot\cos y+\cos x\cdot\sin y=\cos z,\\ \sin y\cdot\cos z+\cos y\cdot\sin z=\cos x,\\ \sin z\cdot\cos x+\cos z\cdot\sin x=\cos y.\end{cases}$$

Using relation A, we obtain the system of equations:

$$\begin{cases}4k\cdot\sin x+5k\cdot\sin y=3k,\\ 3k\cdot\sin y+4k\cdot\sin z=5k,\\ 5k\cdot\sin z+3k\cdot\sin x=4k.\end{cases}$$ We divide each equation of the system into k:

$$\begin{cases}4\sin x+5\sin y=3,\\ 3\sin y+4\sin z=5,\\ 5\sin z+3\sin x=4.\end{cases}$$ We denote $$\begin{matrix}\sin x=a,\\ \sin y=b,\\ \sin z=c.\end{matrix}$$

$$\begin{cases}4a+5b=3,\\ 3b+4c=5,\\ 5c+3a=4.\end{cases}\quad\begin{cases}a=\frac{3-5b}{4},\\ 3b+4c=5,\\ 5c+3\cdot\frac{3-5b}{4}=4\ \big|\cdot 4\end{cases}\quad\begin{cases}3b+4c=5,\\ 20c+9-15b=16.\end{cases}\quad\begin{cases}3b+4c=5\ \big|\cdot 5\\ 20c-15b=7.\end{cases}$$

$$+\begin{cases}15b+20c=25,\\ -15b+20c=7.\end{cases}$$
$$40c=32.$$

$c=\frac{32}{40}=\frac{8}{10}=0{,}8;\ 15b+20\cdot 0{,}8=25;\ 15b=25-16=9,\ b=\frac{9}{15}=0{,}6.$

$a=\frac{3-5\cdot 0{,}6}{4}=\frac{3-3}{4}=0.$

$$\begin{cases}\sin x = 0, & x = \pi n,\ n \in Z.\\ \sin y = 0{,}6, & y = (-1)^m \arcsin 0{,}6 + \pi m,\ m \in Z.\\ \sin z = 0{,}8. & z = (-1)^k \arcsin 0{,}8 + \pi k,\ k \in Z.\end{cases}$$

Answer: $\left(\pi n;\ (-1)^m \arcsin 0{,}6 + \pi m;\ (-1)^k \arcsin 0{,}8 + k\pi\right),\ n \in Z,\quad m \in Z,\quad k \in Z.$

$$\begin{cases}\cos x \cdot \cos 2y + \sin y \cdot \cos 2x + 2\cos x = 1, & (1)\\ \cos 2x + 3\cos 2y + 8\sin y = 8 + 4\sin x \cdot \cos y. & (2)\end{cases}$$

Solution:

Perform some identity transformations of the equation (2):

$\cos^2 x - \sin^2 x + 3 \cdot \left(\cos^2 y - \sin^2 y\right) + 8\sin y = 8 + 4\sin x \cdot \cos y;$

$1 - \sin^2 x - \sin^2 x + 3\left(\cos^2 y - 1 + \cos^2 y\right) + 8\sin y = 8 + 4\sin x \cdot \cos y;$

$1 - 2\sin^2 x + 6\cos^2 y - 3 + 8\sin y = 8 + 4\sin x \cdot \cos y;$

$-2\sin^2 x - 4\sin x \cdot \cos y - 2\cos^2 y + 8\cos^2 y + 8\sin x - 10 = 0 \ \big| \cdot (-1)$

$2\sin^2 x + 4\sin x \cdot \cos y + 2\cos^2 y - 8\cos^2 y - 8\sin x + 10 = 0;$

$2\left(\sin^2 x + 2\sin x \cdot \cos y + \cos^2 y\right) - 8\left(1 - \sin^2 y\right) - 8\sin x + 10 = 0;$

$2 \cdot (\sin x + \cos y)^2 - 8 + 8\sin^2 y - 8\sin x + 10 = 0 \big| : 2$

$(\sin x + \cos y)^2 + 4\sin^2 y - 4\sin y + 1 = 0;$

$(\sin x + \cos y)^2 + (2\sin y - 1)^2 = 0;$

This equality is possible only if $\sin x + \cos y = 0$ (3) and $2\sin y - 1 = 0 \to \sin y = \frac{1}{2}.$

We transform the equation (1):

$\cos x \cdot \left(\cos^2 y - \sin^2 y\right) + \sin y \cdot \left(\cos^2 x - \sin^2 x\right) + 2\cos x = 1;$

$\cos x \cdot \left(1 - \sin^2 y - \sin^2 y\right) + \sin y \cdot \left(\cos^2 x - 1 + \cos^2 x\right) + 2\cos x = 1;$

$\cos x \cdot \left(1 - 2\sin^2 y\right) + \sin y \cdot \left(2\cos^2 x - 1\right) + 2\cos x = 1;$

Substitute in this equation $\sin y = \frac{1}{2}$:

$\cos x \cdot \left(1 - 2 \cdot \left(\frac{1}{2}\right)^2\right) + \frac{1}{2} \cdot \left(2\cos^2 x - 1\right) + 2\cos x = 1;$

$\cos x \cdot \left(1 - \frac{1}{2}\right) + \frac{1}{2} \cdot \left(2\cos^2 x - 1\right) + 2\cos x - 1 = 0;$

$\frac{1}{2}\cos x + \frac{1}{2}\left(2\cos^2 x - 1\right) + 2\cos x - 1 = 0;$

$\cos^2 x + 2{,}5\cos x - 1{,}5 = 0;\ \ |\cos x| \le 1.$

$D = 6{,}25 + 6 = 12{,}25;\ \ \cos x = \frac{-2{,}5 - 3{,}5}{2} = -3$ does not satisfy the condition $|\cos x| \le 1.$

$\cos x = \frac{-2{,}5 + 3{,}5}{2} = \frac{1}{2};\ \ x = \frac{\pi}{3} + 2k\pi;\ \ x = -\frac{\pi}{3} + 2k\pi.$

From the found values, select those that satisfy the equation (3):

$x = \frac{\pi}{3} + 2\pi n,\ \ y = \frac{5\pi}{6} + 2\pi n,\ \ x = -\frac{\pi}{3} + 2k\pi,\ \ y = \frac{\pi}{6} + 2k\pi.$

Answer: $\left(\frac{\pi}{3} + 2\pi n;\ \frac{5\pi}{6} + 2\pi n\right),\ \left(-\frac{\pi}{3} + 2k\pi;\ \frac{\pi}{6} + 2k\pi\right),\ n \in Z,\ k \in Z.$

Self-study assignments:

$$\begin{cases} x-y=-\dfrac{\pi}{6}, \\ \sin x\cdot\sin y=\dfrac{\sqrt{3}}{4}. \end{cases}$$ Answer: $\left(\dfrac{\pi}{6}+\dfrac{k\pi}{2};\ \dfrac{\pi}{3}+\dfrac{k\pi}{2}\right),\ k\in Z.$

$$\begin{cases} x+y=\dfrac{3}{4}, \\ tg\pi x-tg\pi y=2. \end{cases}$$ Answer: $\left(k+\dfrac{5}{12};\ -k+\dfrac{1}{3}\right),\ \left(k+\dfrac{1}{12};\ -k+\dfrac{2}{3}\right).$

$$\begin{cases} x+y=\dfrac{\pi}{3}, \\ tgx\cdot tgy=\dfrac{1}{3}. \end{cases}$$ Answer: $\left(\dfrac{\pi}{6}+k\pi;\ \dfrac{\pi}{6}-k\pi\right).$

$$\begin{cases} 3\cos x+\cos y=2, \\ \sin y=5\sin x. \end{cases}$$ Answer: $(2k\pi;\ \pi+2n\pi),\ k\in Z,\ n\in Z.$

$$\begin{cases} \sin x:\sin y:\sin z=4:3:5, \\ x+y+z=\pi. \end{cases}$$ Answer:

$\left(\left(\dfrac{\pi}{2}-\arccos\dfrac{4}{5};\ 2k+n\right)\pi,\ 2k\pi-\arccos\dfrac{4}{5};\ \pi n+\dfrac{\pi}{2}\right),\ k\in Z,\ n\in Z.$

$$\begin{cases} \sin x\cdot\cos 2y+\sin y\cdot\cos 2x+\sin y=1, \\ 2\cos 2x+8\cos x\cdot\cos y+7=4\sin y. \end{cases}$$ Answer:

$\left((-1)^n\dfrac{\pi}{6}+\pi n;(2k-n+1)\pi-(-1)^n\cdot\dfrac{\pi}{6}\right),\ k\in Z\ \ n\in Z.$

$$\begin{cases} \cos x\cdot\sqrt{\cos 2x}=0, \\ 2\sin^2 x-\cos\left(2y-\dfrac{\pi}{3}\right)=0. \end{cases}$$ Answer: $\left(\dfrac{\pi}{4}(2k+1);\ \dfrac{\pi}{6}(6m+1)\right),\ k,m\in Z.$

$$\begin{cases} x+y=\dfrac{\pi}{6}, \\ 5(\sin 2x+\sin 2y)=2\cdot\left(1+\cos^2(x-y)\right). \end{cases}$$ Answer:

$\left(\dfrac{\pi}{4}(4k+1);\ -\dfrac{\pi}{12}(12k+1)\right),\ \left(\dfrac{\pi}{12}(12k-1);\ \dfrac{\pi}{4}(1-4k)\right),\ k\in Z.$

$$\begin{cases} x-y=\dfrac{5\pi}{3}, \\ \sin 2x=2\sin y. \end{cases}$$ Answer: $\left(\dfrac{\pi}{2}(2k+1);\ \dfrac{\pi}{6}(6k-1)\right),\ k\in Z.$

$$\begin{cases} \sin x\cdot\cos y=\dfrac{1}{4}, \\ \sin y\cdot\cos x=\dfrac{3}{4}. \end{cases}$$ Answer:

$\left(\dfrac{\pi}{6}+\pi(k-m);\ \dfrac{\pi}{3}+\pi(k+m)\right),\ \left(-\dfrac{\pi}{6}+\pi(k-m);\ \dfrac{2\pi}{3}+\pi(k+m)\right),\ k,m\in Z.$

$$\begin{cases} x - y = -\frac{1}{3}, \\ \cos^2 \pi x - \sin^2 \pi x = \frac{1}{2}. \end{cases}$$
Answer: $\left(\frac{6k-1}{6};\ \frac{6k+1}{6} \right),\ k \in Z.$

$$\begin{cases} x + y = \frac{\pi}{4}, \\ tgx \cdot tgy = \frac{1}{6}. \end{cases}$$
Answer: $\left(arctg\frac{1}{2} + \pi k;\ arctg\frac{1}{3} - \pi k \right),\ k \in Z.$

$$\begin{cases} \sqrt{2} \cdot \sin x = \sin y, \\ \sqrt{2} \cdot \cos x = \sqrt{3} \cdot \cos y. \end{cases}$$
Answer: $\left(\pm\frac{\pi}{6} + \pi k;\ \pm\frac{\pi}{4} + \pi m \right),\ k, m \in Z.$

$4 \cdot \left(3 \cdot \sqrt{4x - x^2} \cdot \sin^2 \frac{x+y}{2} + 2\cos(x+y) \right) = 13 + 4\cos^2(x+y).$ Answer: $\left(2;\ \pm\frac{2\pi}{3} - 2 + 2\pi k \right),\ k \in Z.$