Section 19
Finding the largest or smallest value of a square trinomial

To find the largest or smallest value of a square trinomial you need:
1) by the sign of the leading coefficient a of the square trinomial, determine the direction of the branches of the parabola (at $a>0$ branches are directed upwards, when $a<0-$ branches down);
2) at $a>0$ the function has the smallest value, for $a<0-$ greatest value;
3) by the formula $m=-\frac{b}{2a}$ find the abscissa of the vertex of the parabola;
4) by the formula $n=y(m)$ find the ordinate of the vertex of the parabola;
5) the value of the ordinate of the vertex of the parabola and will be the largest or smallest value of the square trinomial.

Find the smallest or largest value of a function $y=x^2-6x+5$.

Solution:

$a=1,\ 1>0,$ and therefore the branches of the parabola are directed upwards - the function has the smallest value.

$m=-\frac{b}{2a}=-\frac{-6}{2\cdot 1}=3;-$ abscissa of the vertex of the parabola.

$n=y(3)=3^2-6\cdot 3+5=9-18+5=-4-$ parabola ordinate.

Smallest function value $y_{найм}=-4.$

Answer: -4.

$y=-x^2-6x+1.$

Solution:

$a=-1,\ -1<0,$ the branches of the parabola are directed downward - the function has the greatest value.

$m=-\frac{b}{2a}=-\frac{6}{2\cdot(-1)}=-3;\ n=y(-3)=-(-3)^2-6\cdot(-3)+1=-9+18+1=10.$

$(-3;10)-$ apex of a parabola.

Largest function value $y_{найб}=10.$

In our opinion, this method is somewhat more rational than the method of isolating the square of the binomial.

Also noteworthy is this way of solving exercises to find the largest or smallest value of a function.

$y=\frac{x^2+x+1}{x^2+2x+1}.$

Solution:

Using the main property of proportion, we get:

$y\cdot\left(x^2+2x+1\right)=x^2+x+1,$
$yx^2+2xy+y-x^2-x-1=0,$
$\left(yx^2-x^2\right)+\left(2xy-x\right)+\left(y-1\right)=0,$
$\left(y-1\right)\cdot x^2+\left(2y-1\right)\cdot x+y-1=0.$

This equation has roots when $D\geq 0,\ D=b^2-4ac,$

$D=\left(2y-1\right)^2-4\cdot\left(y-1\right)\cdot\left(y-1\right)=4y^2-4y+1-y^2\cdot 4+4y+4y-4=4y-3;\ \ 4y-3\geq 0,$

$4y\geq 3,\ \ y\geq\frac{3}{4}.$

The function has the smallest value equal to $\frac{3}{4}$.

Answer: $\frac{3}{4}$.

$f\left(x\right)=\frac{8}{x^2+4\pi x+41}+\cos x.$

Solution:

Consider the denominator of the fraction $x^2+4\pi x+41,\ \ a=1,\ \ 1>0,$ branches are directed upwards, this trinomial has the smallest value, and the fraction $\frac{8}{x^2+4\pi x+41}$ matters the most.

Function $f\left(x\right)=\frac{8}{x^2+4\pi x+41}+\cos x$ has the greatest value, which increases when $\cos x$ It takes its maximum value.

$m=-\frac{b}{2a};\ \ m=-\frac{4\pi}{2\cdot 1}=-2\pi-$ abscissa of the vertex of the parabola.

$y\left(m\right)=y\left(-2\pi\right)=\left(-2\pi\right)^2+4\pi\cdot\left(-2\pi\right)+41=4\pi^2-8\pi^2+41=41-4\pi^2.$

Smallest function value $\varphi\left(x\right)=x^2-4\pi x+41$ is equal to $41-\pi n^2$.

It is also the largest value of the fraction $\frac{8}{x^2+4\pi x+41}$.

Largest function value $y=\cos x$ equals 1.

So, the largest value of the function $f\left(x\right)$ at $x=-2\pi.$

$f\left(-2\pi\right)=\frac{8}{\left(-2\pi\right)^2+4\pi\left(-2\pi\right)+41}+\cos\left(-2\pi\right)=\frac{8}{4\pi^2-8\pi^2+41}+\cos 2\pi=\frac{8}{41-4\pi^2}+1=$

$=\frac{8+41-4\pi^2}{41-4\pi^2}=\frac{49-4\pi^2}{41-4\pi^2}.$

Answer: $\frac{49-4\pi^2}{41-4\pi^2}$.

Find the largest value of a function $f\left(\alpha\right)=\frac{1}{\sin^2\alpha+\cos^2\alpha}$ at $0\leq\alpha\leq 2\pi.$

Solution:

We transform the expression $\sin^2\alpha+\cos^2\alpha$:

You can apply the formula for the sum of cubes in this form:

$a^3+b^3=\left(a+b\right)^3-3ab\left(a+b\right).$

$$\sin^2\alpha+\cos^2\alpha=\left(\sin^2\alpha\right)^3+\left(\cos^2\alpha\right)^3=\left(\underbrace{\sin^2\alpha+\cos^2\alpha}_{1}\right)^3-3\sin^2\alpha\cdot\cos^2\alpha\cdot\left(\sin^2\alpha+\cos^2\alpha\right)=$$

$$=1-3\sin^2\alpha\cdot\cos^2\alpha\cdot 1=1-3\cdot\frac{2\sin\alpha\cdot\cos\alpha\cdot 2\sin\alpha\cdot\cos\alpha}{2\cdot 2}=1-\frac{3}{4}(\sin 2\alpha)^2=1-\frac{3}{4}\cdot\sin^2 2\alpha=$$

$$=1-\frac{3}{4}\cdot\frac{1-\cos 4\alpha}{2}=1-\frac{3}{4}\cdot\left(\frac{1}{2}-\frac{\cos 4\alpha}{2}\right)=1-\frac{3}{8}+\frac{3}{8}\cos 4\alpha=\frac{5}{8}+\frac{3}{8}\cos 4\alpha=\frac{5+3\cos 4\alpha}{8}.$$

Considering the beginning and end of this mathematical phrase, we see that

$$\frac{1}{\sin^6\alpha+\cos^6\alpha}=\frac{8}{5+3\cos 4\alpha}.$$

The last expression is convenient for analysis. It gets the highest value when the denominator $5+\cos 4x$ the smallest value, that is, at $\cos 4\alpha=-1,\ 4\alpha=\pi+2\pi n;$

$$f(\alpha)=\frac{8}{5+3\cdot(-1)}=\frac{8}{5-3}=\frac{8}{2}=4.\ \ \alpha=\frac{\pi}{4}+\frac{\pi n}{2},\ n\in Z.$$

Answer: 4.

The idea of finding the largest or smallest value of a square trinomial can be used to solve equations like: $2^{2\lg x}-2\sin y+2{,}5=2^{\lg x+1}+\frac{1}{2}\cos 2y.$

Solution:

$$2^{2\lg x}-2^{2\lg x+1}=\frac{1}{2}\cos 2y+2\sin y-2{,}5;$$

$$(2\lg x)^2-2\cdot 2^{\lg x}=\frac{1}{2}\left(\cos^2 y-\sin^2 y\right)+2\sin y-2{,}5;$$

$$\underbrace{\left(2^{\lg x}\right)^2-2\cdot 2^{\lg x}+1}_{1}-1=\frac{1}{2}\left(1-\sin^2 y-\sin^2 y\right)+2\sin y-2{,}5;$$

$$\left(2^{\lg x}-1\right)^2-1=\frac{1}{2}\left(1-2\sin^2 y\right)+2\sin y-2{,}5;$$

$$\left(2^{\lg x}-1\right)^2=\frac{1}{2}-\sin^2 y+2\sin y-2{,}5+1;$$

$$\left(2^{\lg x}-1\right)^2=-\left(\sin^2 y+2\sin y+1\right);$$

$$\left(2^{\lg x}-1\right)^2=-(\sin y-1)^2;$$

$$\left(2^{\lg x}-1\right)^2+(\sin y-1)^2=0;$$

The left side of the equation is equal to zero when each of the terms is equal to zero:

$$\left(2^{\lg x}-1\right)^2=0;\ \ 2^{\lg x}-1=0;\ \ 2^{\lg x}=1;\ \ 2^{\lg x}=2^0;\ \ \lg x=0;\ \ x=1.$$

$$(\sin y-1)^2=0;\ \ \sin y=1;\ \ y=\frac{\pi}{2}+2\pi n,\ n\in Z.$$

Answer: $\left(1;\ \frac{\pi}{2}+2k\pi\right), k\in Z.$

Task. From a rectangular sheet of tin with a width of l, it is necessary to bend a gutter with the largest area of a rectangular section. Determine the height of the gutter sides.

Solution:

Let the $AB = x$, then $CD = x$, a $BC = l - 2x$.
The rectangular section of the gutter has an area $S = AB \cdot BC$, $S = x \cdot (l - 2x)$.

$S = lx - 2x^2 = -2x^2 + lx.$

$S = -2x^2 + lx -$ quadratic function.

The graph is a parabola with branches pointing down.

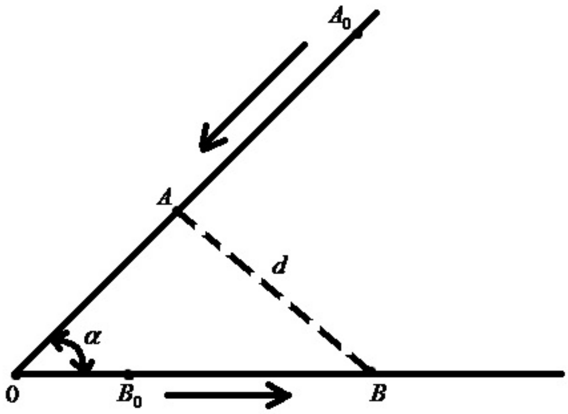
Therefore the function $S = -2x^2 + lx$ matters the most.

$m = -\frac{b}{2a}$; $m = -\frac{l}{-2 \cdot 2} = \frac{l}{4}$; $x = \frac{l}{4} -$ board height.

Answer: $\frac{l}{4}$.

Two points A and B move along straight lines that intersect and form an angle $\alpha = \frac{\pi}{3}$. At the initial moment of time $t = 0$ points occupied position A_0 and B_0.
$A_0O = 8$ м, $B_0O = 2$ м. The movement is shown by arrows.
In how many seconds the distance *d* between points A and B will be the smallest if the speeds of the points $\vartheta_A = 2\, m/s$, $\vartheta_B = 3\, m/s$.
Find the smallest distance between points A and B.

Solution:



Let passed *t* (s) from the beginning of the movement of points. Then the way
$A_0A = \vartheta_A \cdot t = 2 \cdot t$ (м). $B_0B = \vartheta_B \cdot t = 3 \cdot t$ (м).

Way $AO = A_0O - A_0A = (8 - 2t)m$.

$BO = OB_0 + B_0B = (2 + 3t)m$.

Apply the cosine theorem to the triangle *AOB*:

$$AB^2 = OA^2 + OB^2 - 2 \cdot OA \cdot OB \cdot \cos\angle AOB;$$

$$d^2 = (8 - 2t)^2 + (2 + 3t)^2 - 2 \cdot (8 - 2t) \cdot (2 + 3t)\cos\frac{\pi}{3} =$$

$$= 64 - 32t + 4t^2 + 4 + 12t + 9t^2 - (32 + 48t + 8t + 12t^2) \cdot \frac{1}{2} = 19t^2 - 40t + 52.$$

$19 > 0$, and therefore the function $d^2 = 19t^2 - 40t + 52$ takes the smallest value.

$$m = -\frac{b}{2a} = \frac{20}{9}; \; t = \frac{20}{19} s.$$

$$n = d^2(m) = \frac{19}{1} \cdot \left(\frac{20}{19}\right)^2 - \frac{40}{1} \cdot \frac{20}{19} + 52 = \frac{400}{19} - \frac{800}{19} + 52 = -\frac{400}{19} + \frac{52}{1} = \frac{-400 + 988}{19} = \frac{588}{19};$$

$$d_{найм} = \sqrt{\frac{588}{19}}\,(м) \approx 5{,}6\,(m).$$

Answer: $1\frac{1}{19}\,s,\ \approx 5{,}6\,(m).$

Find all pairs of numbers *(x* and *y)*, which satisfy the equation:

$16\sin^2 x\cdot\sin^2 y+tg^2x\cdot tg^2y+18=24\cdot\sin x\cdot\sin y+6tgx\cdot tgy.$

Solution:

$16\sin^2 x\cdot\sin^2 y-24\cdot\sin x\cdot\sin y+3^2+tg^2x\cdot tg^2y-6tgx\cdot tgy+9=0;$

$(4\sin x\cdot\sin y-3)^2+(tgx\cdot tgy-3)^2=0.$

This equality is possible only when:

<table>
<tr>
<td>$(4\sin x\cdot\sin y-3)^2=0;$
$4\sin x\cdot\sin y-3=0;$
$\sin x\cdot\sin y=\frac{3}{4};$
Let us solve the following system of equations:
$+\begin{cases}\cos x\cdot\cos y=\frac{1}{4},\\ \sin x\cdot\sin y=\frac{3}{4}.\end{cases}$
$\cos x\cdot\cos y+\sin x\cdot\sin y=1;$
$\cos(x-y)=1.$
$x-y=2\pi m,\ m\in Z.$
$+\begin{cases}x-y=2\pi m,\\ x+y=\pm\frac{2}{3}\pi+2\pi n.\end{cases}$
$2x=\pm\frac{2}{3}\pi m+2\pi n\ \ |:2$
$x=\pm\frac{\pi}{3}+\pi(m+n), m,n\in Z.$</td>
<td>$(tgx\cdot tgy-3)^2=0;$
$tgx\cdot tgy-3=0;$
$\frac{\sin x}{\cos x}\cdot\frac{\sin y}{\cos y}=3;$
$\cos x\cdot\cos y=\frac{\sin x\cdot\sin y}{3};$
$\cos x\cdot\cos y=\frac{\frac{3}{4}}{3}=\frac{1}{4};$
$-\begin{cases}\cos x\cdot\cos y=\frac{1}{4},\\ \sin x\cdot\sin y=\frac{3}{4}.\end{cases}$
$\cos x\cdot\cos y-\sin x\cdot\sin y=-\frac{1}{2};$
$\cos(x+y)=-\frac{1}{2}.$
$x+y=\pm\frac{2}{3}\pi+2\pi n,\ n\in Z.$
$-\begin{cases}x-y=2\pi n,\\ x+y=\pm\frac{2}{3}\pi+2\pi n.\end{cases}$
$-2y=2\pi n-2m\pi\pm\frac{2\pi}{3}\ \ |:(-2)$
$x=\pi(n-m)\pm\frac{\pi}{3}, n,m\in Z.$</td>
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Answer: $\left(\pm\frac{\pi}{3}+\pi(m+n);\ \pi(n-m)\pm\frac{\pi}{3}\right), n,m\in Z.$

$$\left(x^2+\frac{16}{x^2}\right)\cdot\left(1+\sin^2(x+y)\right)=1+7\cos^2(x+y).$$

Solution:

Let's use the ratio between the arithmetic mean and the geometric mean of the two expressions:

$$\frac{x^2+\dfrac{16}{x^2}}{2}\ge\sqrt{x^2\cdot\frac{16}{x^2}};\ \ \frac{x^4+16}{2x^2}\ge\sqrt{16};\ \ \frac{x^4+16}{2x^2}\ge 4;\ \ x^4-8x^2+16\ge 0.$$

$y(x)=x^4-8x^2+16=\left(x^2-4\right)^2$. The function has the smallest value 0 for $x^2-4=0$,

$$(x-2)\cdot(x+2)=0,\ \left[\begin{array}{l}x-2=0,\\x+2=0.\end{array}\right.\ \left[\begin{array}{l}x=2,\\x=-2.\end{array}\right.$$

For these values of x, the expression $x^2+\dfrac{16}{x^2}=2^2+\dfrac{16}{2^2}=4+4=8.$ $1+\sin^2(x+y)\ge 1.$

So the left side of the equality $\ge 8,$ that is, its smallest value is 8.

Right part $1+7\cos^2(x+y)\le 8.$ Therefore, the equation is equivalent to the system:

$$\begin{cases}x=\pm 2,\\ \sin^2(x+y)=0,\\ \cos^2(x+y)=1.\end{cases}$$ This system is equivalent to a set of systems:

$$\left[\begin{array}{l}\begin{cases}x=2,\\ \sin^2(x+y)=0,\\ \cos^2(x+y)=1.\end{cases}\ \begin{cases}x=2,\\ x+y=\pi n\end{cases}\ \begin{cases}x=2,\\ 2+y=\pi n\end{cases}\ \begin{cases}x=2,\\ y=-2+\pi n.\end{cases}\\ \begin{cases}x=-2,\\ \sin^2(x+y)=0,\\ \cos^2(x+y)=1.\end{cases}\ \begin{cases}x=-2,\\ x+y=\pi n\end{cases}\ \begin{cases}x=-2,\\ -2+y=\pi n\end{cases}\ \begin{cases}x=-2,\\ y=2+\pi n.\end{cases}\end{array}\right.$$

Answer: $\{(2;\ -2+\pi n),\ (2;\ 2+\pi n)\},\ n\in Z.$

$$\cos^4 x+\sin^4 x-\frac{3}{2}\sin^2 2x=-2y^2+4y-3.$$

Solution:

$$\left(\cos^2 x+\sin^2 x\right)^2-2\sin^2 x\cdot\cos^2 x-\frac{3}{2}\sin^2 2x=-2\left(y^2-2y+1{,}5\right);$$

$$1-\frac{1}{2}\sin^2 2x-\frac{3}{2}\sin^2 2x=-2\cdot\left(y^2-2y+1+0{,}5\right);$$

$$1-2\sin^2 2x=-2\cdot(y-1)^2-1,\ \ 2-2\sin^2 2x-2(y-1)^2=0,$$

$$2\left(1-\sin^2 x\right)+2(y-1)^2=0,\ \ 2\cos^2 x+2(y-1)^2=0\ \ |:2\ \ \cos^2 x+(y-1)^2=0.$$

$\cos x=0,$

$y-1=0,$

$y=1.$

$x=\dfrac{\pi}{2}+\pi n.$

Answer: $\left(\dfrac{\pi}{2}+\pi n;\ 1\right).$

$$\cos\left(\frac{3\pi}{2}+2x\right)\cdot\cos x-\cos\left(\frac{\pi}{2}+x\right)\cdot\cos 2x=-2y^2+12y-19.$$

Solution:

$$\sin 2x\cdot\cos x-(-\sin x)\cdot\cos 2x=-2\left(y^2-6y+9{,}5\right);$$

$$\sin 2x\cdot\cos x+\sin x\cdot\cos 2x=-2\left(\left(y^2-6y+9\right)+0{,}5\right);$$

$$\sin(2x+x)=-2\left((y-3)^2-1\right);$$
$$\sin 3x+1+2(y-3)^2=0;$$
$$\frac{2tg\frac{3x}{2}}{1+tg^2\frac{3x}{2}}+1+2(y-3)^2=0;$$
$$\frac{2tg\frac{3x}{2}+1+tg^2\frac{3x}{2}}{1+tg^2\frac{3x}{2}}+2(y-3)^2=0;$$
$$\frac{\left(tg\frac{3x}{2}+1\right)^2}{1+tg^2\frac{3x}{2}}+2(y-3)^2=0,\quad \left[\begin{array}{l}\left(tg\frac{3x}{2}+1\right)^2=0,\\ 2(y-3)^2=0.\end{array}\right.\quad \left[\begin{array}{l}tg\frac{3x}{2}+1=0,\\ y-3=0.\end{array}\right.$$
$$\left[\begin{array}{l}tg\frac{3x}{2}=-1,\\ y=3.\end{array}\right.\quad \left[\begin{array}{l}\frac{3}{2}x=-arctg1+\pi n,\\ y=3.\end{array}\right.\quad \left[\begin{array}{l}\frac{3}{2}x=-\frac{\pi}{4}+\pi n,\\ y=3.\end{array}\right.$$
$$x=-\frac{\pi}{4}\cdot\frac{2}{3}+\frac{2}{3}\pi n;\quad x=-\frac{\pi}{6}+\frac{2}{3}\pi n.$$

Answer: $\left(-\frac{\pi}{6}+\frac{2}{3}\pi n;\ 3\right),\ n\in Z.$

$$\frac{2tg\frac{x}{2}}{1+tg^2\frac{x}{2}}=y^2-4y+5.$$

Solution:

$$\frac{2tg\frac{x}{2}}{1+tg^2\frac{x}{2}}=\left(y^2-4y+4\right)+1;\quad \frac{2tg\frac{x}{2}}{1+tg^2\frac{x}{2}}-1-(y-2)^2=0;\quad \frac{2tg\frac{x}{2}-1-tg^2\frac{x}{2}}{1+tg^2\frac{x}{2}}-(y-2)^2=0;$$
$$\frac{-\left(tg\frac{x}{2}-1\right)^2}{1+tg^2\frac{x}{2}}-(y-2)^2=0;\quad \frac{-\left(tg\frac{x}{2}-1\right)^2}{1+tg^2\frac{x}{2}}-(y-2)^2=0\ \Big|\cdot(-1);\quad \frac{\left(tg\frac{x}{2}-1\right)^2}{1+tg^2\frac{x}{2}}+(y-2)^2=0;$$
$$tg\frac{x}{2}-1=0;\quad tg\frac{x}{2}=\frac{\pi}{4}+\pi n;\quad x=\frac{\pi}{2}+2\pi n,\ n\in Z.$$

Answer: $\left(\frac{\pi}{2}+2\pi n;\ 2\right),\ n\in Z.$

To solve an equation of the second degree in two variables, it is advisable to use the following algorithm:

1) reduce the left and right sides of the equation to the square of the binomial, making sure that the different parts of the equation have opposite signs;

2) collect all the terms of the equation on the left side, and make the right side zero;

3) based on the truth, when the sum of squares is equal to zero, then each of the terms is equal to zero, find all the values of the variables that satisfy the original equation.

$$\frac{1-tg^2\frac{x}{2}}{1+tg^2\frac{x}{2}}=2y^2-4y+3.$$

Solution:

$$\frac{1-tg^2\frac{x}{2}}{1+tg^2\frac{x}{2}}=2\left(y^2-4y+1{,}5\right),\quad \frac{1-tg^2\frac{x}{2}}{1+tg^2\frac{x}{2}}=2(y-1)^2+1;\quad \frac{1-tg^2\frac{x}{2}}{1+tg^2\frac{x}{2}}-1-2(y-1)^2=0;$$

$$\frac{1-tg^2\frac{x}{2}-1-tg^2\frac{x}{2}}{1+tg^2\frac{x}{2}}-2(y-1)^2=0;\quad \frac{-2tg^2\frac{x}{2}}{1+tg^2\frac{x}{2}}-2(y-1)^2=0\ \Big|:(-2);\quad \frac{tg^2\frac{x}{2}}{1+tg^2\frac{x}{2}}+(y-1)^2=0;$$

$$\frac{tg^2\frac{x}{2}}{1+tg^2\frac{x}{2}}=0;\quad tg^2\frac{x}{2}=0,\quad tg\frac{x}{2}=0,\quad \frac{x}{2}=\pi n,\quad x=2\pi n.$$

Answer: $(2\pi n;\ 1)$.

Self-study assignments:

Find smallest function value $y=x^2-8x+12.$

Answer: $-4.$

Find the largest value of a function $y=-x^2-8x+20.$

Answer: $4.$

Find the largest value of an expression $\frac{tg\alpha-tg\alpha}{\cos 4\alpha+1}$ at $0<\alpha<\frac{\pi}{4}.$

Answer: 2 at $\alpha=\frac{\pi}{8}.$

Find all values of *x* and *y* that satisfy the equation

$$\cos^4 x+\sin^4 x-\frac{3}{2}\sin^2 2x=-2y^2+4y-3.$$

Answer: $\left(\frac{\pi}{2}+\pi n;\ 1\right),\ n\in Z.$

$$\frac{2tg\frac{3}{2}x}{1+tg^2\frac{3}{2}x}=2y^2-8y+9.$$

Answer: $\left(\frac{\pi}{6}+\frac{2\pi n}{3};\ 2\right),\ n\in Z.$

Of all the rectangles of a given perimeter *2P*, find the one with the largest area.

Answer: *square with side* $\frac{P}{2}$.

With a wire of length *l* cm, it is necessary to make a model of a rectangular parallelepiped of the largest volume. Determine the lengths of the edges of a parallelepiped.

Answer: *cube with edge* $\frac{l}{12}$.

Find all values of *x* and *y* that satisfy the equation $\frac{2tg\frac{3}{2}x}{1+tg^2\frac{3}{2}x}=2y^2-8y+9.$

Answer: $\left(\frac{\pi}{6}+\frac{2}{3}\pi n;\ 2\right),\ n\in Z.$