Section 17
Proof of trigonometric inequalities

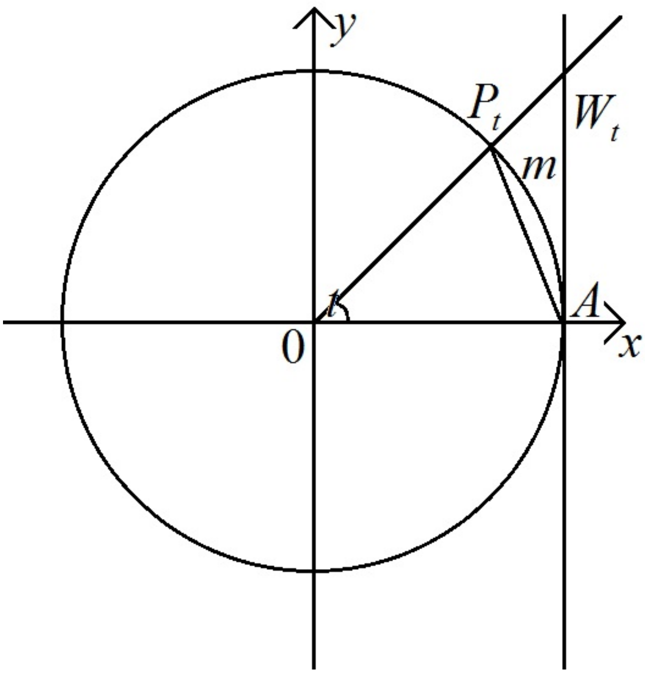
When solving exercises for proving trigonometric inequalities, it is necessary to carry out such transformations as a result of which we arrive at an inequality of the form $|\sin x| \le 1$ or $|\cos x| \le 1$.

Prove that when $0 < t < \frac{\pi}{2}$ the inequalities are true:

$$\sin t < t < tgt,$$

$$\cos t < \frac{\sin t}{t} < 1.$$

Proof:



In the first quarter of the unit circle, choose a point P_t, which corresponds to the real number *t*. Drawing a unit radius at this point OP_t, we get $\angle P_tOA = t$ *(in radians)*. Let's compare the area of the sector P_tOAm and triangle OAW_t:
straight AW_t – tangent axis tangent to a circle at a point A, and therefore $OA \perp AW_r$.

$$S_{OAW_t} = \frac{1}{2}|OA| \cdot |AW_t| = \frac{1}{2} \cdot 1 \cdot |AW_t| = \frac{1}{2} tgt.$$

$$S_{AOP_t} = \frac{1}{2}|OA| \cdot |OP_t| \cdot \sin t = \frac{1}{2} \cdot 1 \cdot 1 \cdot \sin t = \frac{1}{2} \sin t.$$

Since the circle is tangent, then $r = 1$, and its area of a circle $S = \pi r^2 = \pi \cdot 1^2 = \pi.$
It can be considered as the area of a sector in 2π radian. Then such a proportion is possible:

$$\begin{array}{l} 2\pi_{\text{рад}} - \pi \\ t_{\text{рад}} - x \end{array} \quad x = \frac{t \cdot \pi}{2\pi} = \frac{t}{2}.$$ So, the area of the sector $S_{AOMP_t} = \frac{t}{2}.$

The figure shows that $S_{\Delta OAP_t} < S_{OAMP_t} < S_{\Delta OAW_t}$.
Substituting the values of the areas into the double inequality, we obtain the inequality:

$$\frac{1}{2}\sin t < \frac{t}{2} < \frac{1}{2}tgt \ \Big|\cdot 2,$$

$$\sin t < t < tgt \ (A).$$

Since the Sine function in the first coordinate quarter is positive, dividing the last inequality by $\sin t > 0,$ we obtain the correct inequality:

$$\frac{\sin t}{\sin t} < \frac{t}{\sin t} < \frac{tgt}{\sin t};$$

$$1 < \frac{t}{\sin t} < \frac{1}{\cos t} \text{ or}$$

$$\cos t < \frac{\sin t}{t} < 1.(B)$$

Q.E.D.

Prove that $\sin 1 > \frac{\pi}{4}.$

Proof:

According to reduction formulas $\sin 1 = \cos\left(\frac{\pi}{2} - 1\right) = \cos\left(2\cdot\left(\frac{\pi}{4} - \frac{1}{2}\right)\right) = 1 - 2\sin^2\left(\frac{\pi}{4} - \frac{1}{2}\right).$

We use the inequality A:

$\sin\left(\frac{\pi}{2} - \frac{1}{2}\right) < \frac{\pi}{4} - \frac{1}{2}.$ Let's square both sides of the equation:

$\sin^2\left(\frac{\pi}{2} - \frac{1}{2}\right) < \left(\frac{\pi}{4} - \frac{1}{2}\right)^2$, then $1 - 2\sin^2\left(\frac{\pi}{4} - \frac{1}{2}\right) > 1 - 2\left(\frac{\pi}{4} - \frac{1}{2}\right)^2$;

$$1 - 2\cdot\left(\frac{\pi^2}{16} - 2\cdot\frac{\pi}{4}\cdot\frac{1}{2} + 1\right) - 1 - \frac{\pi^2}{8} + \frac{\pi}{2} - \frac{1}{2} = \frac{1}{2} - \frac{\pi^2}{8} + \frac{\pi}{2} = \frac{4 - \pi^2 + 4\pi}{8} = -\frac{(2-\pi)^2}{8} < 0;$$

$1 - 2\sin^2\left(\frac{\pi}{4} - \frac{1}{2}\right) > \frac{\pi}{4}.$ Q.E.D.

Prove that when $0 < t < \frac{\pi}{2}$ correct inequality $t - \frac{t^3}{4} < \sin t.$

Proof:

From the inequality $\frac{t}{2} < \frac{tgt}{2}$ it follows that $\frac{t}{2} < tg\frac{t}{2}\Big|\cdot 2\cos^2\frac{t}{2},\ t\cdot\cos^2\frac{t}{2} < tg\frac{t}{2}\cdot 2\cos^2\frac{t}{2}$

$$\rightarrow t\cos^2\frac{t}{2} < \frac{\sin\frac{t}{2}}{\cos\frac{t}{2}}\cdot 2\cos^2\frac{t}{2},\ t\cdot\cos^2\frac{t}{2} < 2\sin\frac{t}{2}\cdot\cos\frac{t}{2},\ t\cdot\cos^2\frac{t}{2} < \sin t.$$

Insofar as $\sin\frac{t}{2} < \frac{t}{2}$ i $\cos^2\frac{t}{2} = 1 - \sin^2\frac{t}{2},$ то $t\cdot\left(1 - \sin^2\frac{t}{2}\right) < \sin t.$

Replace $\sin\frac{t}{2}$ by $\frac{t}{2}$ (on the left) $t\cdot\left(1 - \frac{t^2}{2}\right) < \sin t,\ t - \frac{t^3}{2} < \sin t,$ Q.E.D.

Prove inequality $\sin\alpha\cdot\sin 2\alpha\cdot\sin 3\alpha<\frac{3}{4}.$

Proof

$$\sin\alpha\cdot\sin 2\alpha\cdot\sin 3\alpha=\sin 2\alpha\cdot(\sin\alpha\cdot\sin 3\alpha)=\sin 2\alpha\cdot\frac{\cos(\alpha-3\alpha)-\cos(\alpha+3\alpha)}{2}=$$

$$=\sin 2\alpha\cdot\frac{\cos\alpha-\cos 4\alpha}{2}=\sin 2\alpha\cdot\frac{\cos 2\alpha-\cos 4\alpha}{2}=\frac{(\sin 2\alpha\cdot\cos 2\alpha-\sin 2\alpha\cdot\cos 4\alpha)\cdot 2}{2\cdot 2}=$$

$$=\frac{2\sin 2\alpha\cdot\cos 2\alpha-2\sin 2\alpha\cdot\cos 4\alpha}{4}=\frac{\sin 4\alpha-2\cdot\frac{\sin(-2\alpha)+\sin 6\alpha}{2}}{4}=\frac{\sin 4\alpha+\sin 2\alpha-\sin 6\alpha}{4}.$$

Function $y=\sin t$ exists at $|\sin t|\le 1.$

If only $\sin 4\alpha=1,$ $\sin 2\alpha=1,$ $-\sin 6\alpha=1,$ then the numerator of the last fraction would be $1+1+1=3,$ and since $\sin 2\alpha$ и $\sin 4\alpha$ simultaneously cannot be equal to one, then this sum $<3.$

Really, $\sin 4\alpha=2\sin 2\alpha\cdot\cos 2\alpha,$ if $\sin 2\alpha=1,$ then $\sin 4\alpha=2\cdot 1\cdot 0=0\ne 1.$

Well then, $\frac{\sin 4\alpha+\sin 2\alpha-\sin 6\alpha}{4}<\frac{3}{4}.$

Prove that inequality $\frac{\cos x}{\sin^2 x(\cos x-\sin x)}>8$ at $0<x<\frac{\pi}{4}.$

Proof

It's obvious that $\cos x\ne 0$ and then $\cos^3 x\ne 0.$

Divide the numerator and denominator of the fraction by $\cos^3 x$:

$$\frac{\frac{\cos x}{\cos^3 x}}{\frac{\sin^2 x}{\cos^2 x}\cdot\frac{\cos x-\sin x}{\cos x}}=\frac{\frac{1}{\cos^2 x}}{tg^2 x\cdot\left(\frac{\cos x}{\cos x}-\frac{\sin x}{\cos x}\right)}=\frac{\frac{1}{\cos^2 x}}{tg^2 x\cdot(1-tgx)}=\frac{1+tg^2 x}{tg^2 x\cdot(1-tgx)}=\frac{(1-tgx)^2+2tgx}{tg^2 x(1-tgx)}=$$

$$=\frac{(1-tgx)^2}{tg^2 x(1-tgx)}+\frac{2tgx}{tg^2 x(1-tgx)}=\frac{1-tgx}{tg^2 x}+\frac{2}{tgx(1-tgx)}\quad(A)$$

In that $0<x<\frac{\pi}{4},$ then $tgx>0$ and $1-tgx>0.$

At $tgx=\frac{1}{2}$ expression value $\frac{2}{tgx(1-tgx)}=\frac{2}{\frac{1}{2}\left(1-\frac{1}{2}\right)}=8.$

Expression value A - positive.

Well then, $\frac{\cos x}{\sin^2 x\cdot(\cos x-\sin x)}>8$ at $0<x<\frac{\pi}{4}.$ Inequality is proved.

Prove inequality $\sin^8 x+\cos^8 x\ge\frac{1}{8}.$ For what values of x is the inequality?

Solution:

By the basic trigonometric identity $\sin^2 x+\cos^2 x=1.$

Let's square both sides of the equation:

$(\sin^2 x+\cos^2 x)=1^2.$

$\sin^4 x+2\sin^2 x\cdot\cos^2 x+\cos^4 x=1;$ $\sin^4 x+\cos^4 x\ge 2\sin^2 x\cdot\cos^2 x;$ $\sin^4 x+\cos^4 x\ge\frac{1}{2};$

$\left(\sin^4 x+\cos^4 x\right)^2=\sin^8 x+2\sin^4 x\cdot\cos^4 x+\cos^8 x\ge\frac{1}{4};\ \sin^8 x+\cos^8 x\ge 2\sin^4 x\cdot\cos^4 x.$

Well then, $\sin^8 x+\cos^8 x\ge\frac{1}{8}.$

The equal sign is attained at $\sin x=\cos x\,|:\cos x,\ \ tgx=1,\ \ x=\frac{\pi}{4}+\frac{\pi n}{2}, n\in Z.$

Prove inequality $(x+y)\cdot(x+y+2\cos x)+2\ge 2\sin^2 x.$
At what values of x and y is the equality?

$(x+y)\cdot(x+y)+(x+y)\cdot 2\cos x+2-2\sin^2 x\ge 0;$

$(x+y)^2+(x+y)2\cos x+2\left(1-\sin^2 x\right)\ge 0;$

$(x+y)^2+2(x+y)\cos x+2\cos^2 x\ge 0;$

$\underbrace{(x+y)^2+2(x+y)\cos x+\cos^2 x}_{квадрат\ двочлена}+\cos^2 x\ge 0;$

$\underbrace{\left((x+y)+\cos x\right)^2}_{\ge 0}+\underbrace{\cos^2 x}_{\ge 0}\ge 0.$

Since both terms are integral, then their sum is also integral. Inequality is proved.

Equality is achieved when: $\begin{cases}x+y+\cos x=0,\\ \cos^2 x=0.\end{cases}\ \begin{cases}x+y=0,\\ \cos x=0.\end{cases}\ \begin{cases}y=-x,\\ \cos x=0.\end{cases}$

Solving the second equation of the system, we obtain: $x=\frac{\pi}{2}+\pi n,\ y=-\frac{\pi}{2}+\pi n, n\in Z.$

Answer: $\left(\frac{\pi}{2}+\pi n; -\frac{\pi}{2}+\pi n\right), n\in Z.$

Prove that $-4\le\cos 2x+3\sin x\le 2\frac{1}{8}.$

Proof

Consider the function $y=\cos 2x+3\sin x.$ Let's simplify it:

$y=\cos 2x+3\sin x=\cos^2 x-\sin^2 x+3\sin x=1-\sin^2 x-\sin^2 x+3\sin x=-2\sin^2 x+3\sin x+1.$

This is a quadratic function with respect to $\sin x.$
Let the $\sin x=t,$ then $y=-2t^2+3t+1,\ \ -1\le t\le 1,\ -2<0$ So the branches of the parabola facing down.

Find the coordinates of the vertex of the parabola: $t_0=\frac{-b}{2a}=\frac{-3}{-2\cdot 2}=\frac{3}{4};\ \frac{3}{4}\in[-1;1]$

$y_0=y(t_0)=-2\left(\frac{3}{4}\right)^2+3\cdot\frac{3}{4}+1=-2\cdot\frac{9}{16}+\frac{9}{4}+1=-\frac{18}{16}+\frac{9}{4}+1=\frac{-18+36+16}{16}=\frac{34}{16}=2\frac{2}{16}=2\frac{1}{8};$

$2\frac{1}{8}-$ highest function value.

Find the value of this function at the ends $[-1;1]$:

$y(-1)=-2\cdot(-1)^2+3\cdot(-1)+1=-2-3+1=-4;$

$y(1)=-2\cdot 1^2+3\cdot 1+1=-2+3+1=2.$

So, the smallest function value on $[-1;1]$ is equal $-4,$ and the largest value $2\frac{1}{8}.$

Thus, $-4\le\cos 2x+3\sin x\le 2\frac{1}{8}.$

Prove that: $0 < \sin^8 x + \cos^{14} x \leq 1.$

Proof

Because $\sin x \leq 1,\ \ \sin^2 x \leq 1,\ \ \cos x \leq 1,\ \ \cos^2 x \leq 1$ for any x.
Then $\sin^8 x \leq \sin^2 x,\ \ \cos^{14} x \leq \cos^2 x.$ Let us add these inequalities term by term:
$\sin^8 x + \cos^{14} x \leq \sin^2 x + \cos^2 x,\ \ \sin^8 x + \cos^{14} x \leq 1.$
Q.E.D.