Section 10
Equations with Modules

$2\cdot|x-4|-6=0.$

Solution:

$2\cdot|x-4|=6;\quad |x-4|=\frac{6}{2};\quad |x-4|=3.$

This equation is equivalent to the aggregate of such:

$\left[\begin{array}{l} x-4=3, \\ x-4=-3 \end{array}\right. \left[\begin{array}{l} x=3+4, \\ x=-3+4 \end{array}\right. \left[\begin{array}{l} x=7, \\ x=1 \end{array}\right.$

Answer: 1; 7.

$|2x+1|+4=0.$

Solution:

$|2x+1|=-4.$ By the properties of the module, this equation has no roots.

Answer: $\varnothing$.

$|3|2x-1|-2|-6=0.$

Solution:

$|3|2x-1|-2|=6.$

This equation is equivalent to such a set of equations:

$\left[\begin{array}{l} |3|2x-1|-2|=6, \\ |3|2x-1|-2|=-6 \end{array}\right. \left[\begin{array}{l} 3|2x-1|=6+2, \\ 3|2x-1|=-6+2 \end{array}\right. \left[\begin{array}{l} 3|2x-1|=8, \\ 3|2x-1|=-4 \end{array}\right.$

$\left[\begin{array}{l} |2x-1|=\frac{8}{3}, \quad \left[\begin{array}{l} 2x-1=\frac{8}{3}, \\ 2x=-\frac{8}{3} \end{array}\right. \left[\begin{array}{l} 2x=2\frac{2}{3}+1, \\ 2x=-2\frac{2}{3}+1 \end{array}\right. \left[\begin{array}{l} 2x=3\frac{2}{3}, \\ 2x=-1\frac{2}{3}. \end{array}\right. \left[\begin{array}{l} x=3\frac{2}{3}:2=\frac{11}{3}\cdot\frac{1}{2}=\frac{11}{6}, \\ x=-1\frac{2}{3}:2=-\frac{5}{6}\cdot\frac{1}{2}=-\frac{5}{6}. \end{array}\right. \\ |2x-1|=-\frac{4}{3} - \text{це рівняння коренів не має.} \end{array}\right.$

Answer: $-\frac{5}{6};\ 1\frac{5}{6}.$

$3\cdot|x-5|+2c=8.$

Solution:

If in a linear equation the modulus sign occurs only once, then it is advisable to solve it below by the proposed method:

1). $|x-5|=x-5$ at $x-5\geq 0,$ that is, when $x\geq 5.$ This equation takes the form:

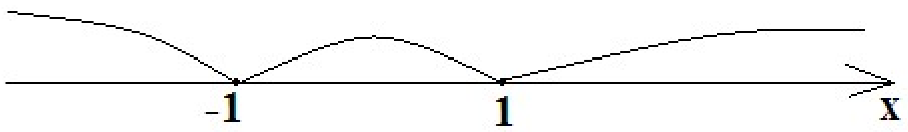
$3(x-5)+2x=8,\ 3x-15+2x=8,\ 5x-15=8,\ 5x=8+15,\ 5x=23,\ x=\frac{23}{5}=4{,}6.$ 4,6 does not satisfy the condition $x\geq 5,$ and therefore 4,6 – extraneous root.

2). $|x-5|=-(x-5)$ at $x-5<0,$ that is, when $x<5.$ $3(-(x-5))+2x=8,\ 3(5-x)+2x=8,$ $15-3x+2x=8,\ 15-x=8;\quad x=15-8=7.$ 7 – does not satisfy the condition $x<5,$ and therefore 7 – extraneous root.

Answer: $\varnothing$.
If the modulus sign appears more than once in a linear equation, then it is convenient to use the general method of solving.
$2|x+1|-|2x-2|=9.$

Solution:

We equate the submodular expressions to zero and solve the resulting equations:
$\left[\begin{array}{l} x+1=0, \\ 2x-2=0. \end{array}\right. \left[\begin{array}{l} x=-1, \\ 2x=2. \end{array}\right. \left[\begin{array}{l} x=-1, \\ x=1. \end{array}\right.$
We denote the found roots of the equations on the number line:



We transform this equation at each of the formed numerical intervals and solve it.
On interval $(-\infty;-1)$ $x+1<0,$ а потому $|x+1|=-(x+1);$ $2x-2<0,$
$|2x-2|=-(2x-2).$
This equation is in this interval:
$2\cdot(-(x+1))-(-(2x-2))=9,$ $-2x-2+2x-2=9;$ $-4=9$, which is impossible, and therefore on this interval the equation of roots does not have. On $[-1;1)$
$x+1>0,$ $|x+1|=x+1.$ $2x-2<0,$ $|2x-2|=2-2x,$ $2(x+1)-(2-2x)=9,$
$2x+2-2+2x=9,$ $4x=9;$ $x=\dfrac{9}{4}=2{,}25\notin[-1;1).$ $x\in\varnothing.$
On $[1;+\infty),$ $x+1>0,$ $|x+1|=x+1.$ $2x-2\geq 0,$ $|2x-2|=2x-2.$ Then
$2(x+1)-(2x-2)=9.$ $2x+2-2x+2=9,$ $4=9,$ which is impossible $x\in\varnothing.$
Answer: $\varnothing$.

$|x-4|+|x-5|=1.$

Solution:

$\left[\begin{array}{l} x-4=0, \\ x-5=0. \end{array}\right. \left[\begin{array}{l} x=4, \\ x=5. \end{array}\right.$
On $(-\infty;4)$ $x-4<0,$ $|x-4|=4-x.$ $x-5<0,$ $|x-5|=5-x.$ $4-x+5-x=1,$
$9-2x=1.$ $-2x=-8,$ $x=4\notin(-\infty;\ 4),$ $x\in\varnothing.$
On $[4;5)$ $x-4\geq 0,$ $|x-4|=x-4.$ $x-5<0,$ $|x-5|=5-x.$
Тогда $x-4+5-x=1,$ $0\cdot x=1-1;$ $0\cdot x=0.$ $x-$ any number, that is, the solution to the equation is the interval $[4;\ 5).$
On $[5;+\infty)$ $x-4>0,$ $|x-4|=x-4.$ $x-5>0,$ $|x-5|=5-x,$ $x-4+x-5=1,$
$2x-9=1,$ $2x=10,$ $x=5\in[5;\ +\infty).$
Answer: $[4;\ 5)\cup\{5\}=[4;\ 5].$
Sometimes it is possible to do without general methods of solving the equation.
For example,

$|3x+7|+|8x-1|+|x-2|+|4-5x|+20x+10=0.$

Solution:

Since at $x \geq 0$ the left side of the equation is positive, then it has only negative roots. At $x<0$ the equation takes the form:

$7-3x+1-8x+2-x+4-5x+20x+10=0,\ 3x+24=0,\ 3x=-24,\ x=-8.$

Answer: -8.

Sometimes results in the success of using the parity of a function, for example:

$|x-1|+|x+1|+|2x+3|+|2x-3|=30.$

Solution:

Consider the function $f(x)=|x-1|+|x+1|+|2x+3|+|2x-3|-30.$

Check for parity:

$f(x)=|-x-1|+|-x+1|+|-2x+3|+|-2x-3|-30=|-(x+1)|+|-(x-1)|+|-(2x-3)|+$
$+|-(2x+3)|-30=|x+1|+|x-1|+|2x-3|+|2x+3|-30=f(x).$

This function is paired. It is known that the nonzero roots of the corresponding equation are opposite, therefore, to solve this equation, it is enough to find its inalienable roots. To do this, consider it only at integral intervals: $[0;1)$; $[1;1{,}5)$ i $[1{,}5;+\infty)$. On $[0;1)$ $x-1<0,\ |x-1|=1-x.$

$$x+1>0,\ |x+1|=x+1.$$
$$2x+3>0,\ |2x+3|=2x+3.$$
$$2x-3<0.\ |2x-3|=3-2x.$$

The equation takes the form:

$1-x+x+1+2x+3+3-2x=30,\ 8=30-$ impossible because $x\in\varnothing$.

On $[1;1{,}5)$

$$x-1>0,\ |x-1|=x-1.$$
$$x+1>0,\ |x+1|=x+1.$$
$$2x+3>0,\ |2x+3|=2x+3.$$
$$2x-3<0.\ |2x-3|=3-2x.$$

$x-1+x+1+2x+3+3-2x=30,\ 2x+6=30,\ 2x=24,\ x=12\notin[1;1{,}5).$ Means, $x\in\varnothing$.

On $[1{,}5;+\infty)$

$$x-1>0,\ |x-1|=x-1.$$
$$x+1>0,\ |x+1|=x+1.$$
$$2x+3>0,\ |2x+3|=2x+3.\ 2x-3>0,$$

$|2x-3|=2x-3.$

$x-1+x+1+2x+3+2x-3=30,\ 6x=360,\ x=5.$ Number -5.

Answer: $-5;\ 5.$

$x^2-6|x|+8=0.$

Solution:

By module properties $x^2=|x|^2$ we have a quadratic equation for $|x|$:

$x^2-6|x|+8=0$. By Vieta's theorem: $|x|=2$ i $|x|=4$.
From here $x_1=-2;\ x_2=2;\ x_3=-4;\ x_4=4$.
Answer: $-4;\ -2;\ 2;\ 4$.

$$\left|\frac{2x^2+5x-7}{x^2+x+4}\right|=\frac{5}{2}.$$

Solution:

$$\left[\begin{array}{l}\dfrac{2x^2+5x-7}{x^2+x+4}=-\dfrac{5}{2},\\ \dfrac{2x^2+5x-7}{x^2+x+4}=\dfrac{5}{2}.\end{array}\right.$$

The discriminant of the quadratic trinomial to the denominator of the fraction on the left side of the equation $D=1-16=-15<0$, то $x^2+x+4\neq 0$.
That is range of valid values = R.
To solve the equations of the set, the main property of the proportion is used:

$$\left[\begin{array}{l}2\cdot\left(2x^2+5x-7\right)=-5\left(x^2+x+4\right),\\ 2\cdot\left(2x^2+5x-7\right)=5\left(x^2+x+4\right).\end{array}\right. \left[\begin{array}{l}4x^2+10x-14=-5x^2-5x-20,\\ 4x^2+10x-14=5x^2+5x+20.\end{array}\right.$$

$$\left[\begin{array}{l}9x^2+15x+6=0,\ \ |:3\\ -x^2+5x-34=0.\ \ |:(-1)\end{array}\right. \left[\begin{array}{l}3x^2+5x+2=0,\ D=25-24=1.\\ x^2-5x+34=0.\ D=25-136=-111<0\ \ x\in\varnothing.\end{array}\right.$$

$$x_1=\frac{-5-1}{6}=-1;\quad x_2=\frac{-5+1}{6}=-\frac{2}{3}.$$

Check: If $x=-1$, then $\left|\dfrac{2(-1)^2+5(-1)-7}{(-1)^2-1+4}\right|=\left|\dfrac{2-5-7}{1-1+4}\right|=\left|\dfrac{-10}{4}\right|=\left|-\dfrac{5}{2}\right|=\dfrac{5}{2};$

If $x=-\dfrac{2}{3}$, then $\left|\dfrac{2\cdot\left(-\frac{2}{3}\right)^2+5\cdot\left(-\frac{2}{3}\right)-7}{\left(-\frac{2}{3}\right)^2-\frac{2}{3}+4}\right|=\left|\dfrac{\frac{8}{9}-\frac{10}{3}-7}{\frac{4}{9}-\frac{2}{3}+4}\right|=\left|\dfrac{8-30-63}{4-6+36}\right|=\left|\dfrac{-85}{34}\right|=$

$\left|-\dfrac{5}{2}\right|=\dfrac{5}{2}.$

Answer: $-\dfrac{2}{3};\ -1.$

$$\left(x^2+2x-2\right)^2-4\cdot\left|x^2+2x-2\right|+3=0.$$

Solution

$$\left(x^2+2x-2\right)^2=\left|x^2+2x-2\right|^2.$$

$$\left|x^2+2x-2\right|^2-4\cdot\left|x^2+2x-2\right|+3=0.$$

We introduce a new variable $\left|x^2+2x-2\right|=t$. Then the equation has the form:

$t^2-4t+3=0,\ t_1=1,\ t_2=3.$

$$\left[\begin{array}{l}|x^2+2x-2|=1,\\ |x^2+2x-2|=3.\end{array}\right.\quad \left[\begin{array}{l}x^2+2x-2=1,\\ x^2+2x-2=-1,\\ x^2+2x-2=3,\\ x^2+2x-2=-3.\end{array}\right.\quad \left[\begin{array}{l}x^2+2x-3=0,\\ x^2+2x-1=0,\\ x^2+2x-5=0,\\ x^2+2x+1=0.\end{array}\right.$$

1). $x^2+2x-3=0,\quad x_1=1,\quad x_2=-3.$

2). $x^2+2x-1=0,\quad x_3=-1-\sqrt{2},\quad x_4=-1+\sqrt{2}.$

3). $x^2+2x-5=0,\quad x_5=-1-\sqrt{6},\quad x_6=-1+\sqrt{6}.$

4). $x^2+2x+1=0,\quad (x+1)^2=0,\quad x+1=0,\quad x_7=-1.$

Answer: $-1-\sqrt{6};\ -1-\sqrt{2};\ -3;\ -1;\ -1+\sqrt{2};\ -1+\sqrt{6};\ 1.$

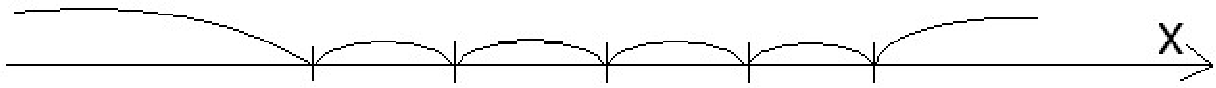
$|x^2-2x-3|+2|2x-3|-6=0.$

Solution:

Let's use the general method:

$x^2-2x-3=0.\quad 2x-3=0,$

$x_1=-1;\ x_2=3.\quad 2x=3,\quad x=1{,}5.$



On $(-\infty;-1]$ $\quad x^2-2x-3\ge 0,\quad |x^2-2x-3|=x^2-2x-3.$

Look. A $\quad 2x-3<0,\quad |2x-3|=3-2x.$

On $(-1;1{,}5]$ $\quad x^2-2x-3<0,\quad |x^2-2x-3|=-x^2+2x+3.$

Look. B $\quad 2x-3<0.\quad |2x-3|=3-2x.$

On $(1{,}5;3]$ $\quad x^2-2x-3<0,\quad |x^2-2x-3|=-x^2+2x+3.$

Look. C $\quad 2x-3>0.\quad |2x-3|=2x-3.$

On $(3;+\infty)$ $\quad x^2-2x-3>0,\quad |x^2-2x-3|=x^2-2x-3.$

Look. D $\quad 2x-3>0.\quad |2x-3|=2x-3.$

A: $x^2-2x-3+2(3-2x)-6=0,\quad x^2-2x-3+6-4x-6=0,\quad x^2-6x-3=0;$

$D=36+12=48=4\cdot 12=16\cdot 3\quad x_1=\dfrac{6-4\sqrt{3}}{2}=3-2\sqrt{3};\quad x_2=3+2\sqrt{3}.$ x_1 and $x_2\notin(-\infty;-1)$. Means, $x\in\varnothing$.

B: $-x^2+2x+3+2(3-2x)-6=0, \quad -x^2+2x+3+6-4x-6=0,$
$-x^2-2x+3=0\big|\times(-1).\ x^2+2x-3=0; \quad x_1=-3; \quad x_2=1.\ -3\notin(-1;1{,}5).$
$1\in(-1;1{,}5). \quad x=1.$
C: $-x^2+2x+3+2\cdot(3-2x)-6=0,\ -x^2+2x+3+4x-6-6=0,\ -x^2+6x-9=0\cdot(-1),$
$x^2-6x+9=0,\ (x-3)^2=0,\ x=3\in(1{,}5;3].$
D: $x^2-2x-3+2\cdot(2x-3)-6=0,\ x^2-2x-3+4x-6-6=0,\ x^2+2x-15=0,\ x_1=-5;$
$x_2=3.\ -5\in(3;+\infty).\ x\in\varnothing.$
Answer: 1; 3.

$\left|x^2-x-3\right|+2\left|x^2+2x-3\right|=9\cdot|x|.$

Solution

An equation of this type can be solved in a simpler way than the traditional one, namely: since $x\neq 0$, then we divide both sides of the equation by $|x|$.

$\left|x-1-\frac{1}{3}\right|+2\left|x+2-\frac{3}{x}\right|=9,$ (A).

We denote $x-1-\frac{1}{3}=y$, then, adding the number 3 to both sides of the last equality, we obtain $x-1-\frac{3}{x}+3=y+3,\ x+2-\frac{3}{x}=y+3.$ Equation A takes the form

$|y|+2|y+3|=9.\ y=0,\ y+3=0,\ y=-3$

On $(-\infty;-3)$ $y<0,\ |y|=-y. \quad y+3<0, \quad |y+3|=-y-3.$
$-y+2\cdot(-y-3)=9, \quad -y-2y-6=9,\ -3y=15,\ y=-5.$
On $[-3;0)$ $y<0,\ |y|=-y. \quad y+3>0, \quad |y+3|=y+3.$
$-y+2\cdot(y+3)=9, \quad -y+2y+6=9, \quad y=3.\ 3\notin[-3;0),\ y\in\varnothing.$
On $[0;+\infty)$ $y>0,\ |y|=y. \quad y+3>0, \quad |y+3|=y+3.$
$y+2\cdot(y+3)=9, \quad y+2y+6=9,\ 3y=3,\ y=1.\ 1\notin[0;+\infty).$
We return to the replacement, we get a set of equations:

$$\left[\begin{array}{l} x-1-\frac{3}{x}=-5, \\ x-1-\frac{3}{x}=1. \end{array}\right. \quad \left[\begin{array}{l} x^2-x-3+5x=0, \\ x^2-x-3-x=0. \end{array}\right. \quad \left[\begin{array}{l} x^2-4x-3=0, \\ x^2-2x-3=0 \end{array}\right.$$

1). $x^2+4x-3=0,\ D=16+12=28=4\cdot 7. \quad x_1=\frac{-4-2\sqrt{7}}{2}=-2-\sqrt{7}; \quad x_2=-2+\sqrt{7}.$
2). Equation roots $x^2-2x-3=0$ we find by Vieta's theorem $x_3=-1,\ x_4=3.$
Answer: $-2-\sqrt{7};\ -1;\ -2+\sqrt{7};\ 3.$
$\left|x^2-4x+7\right|+2|x-2|-6=0.$ (1)

Solution

Since the discriminant of the square trinomial $x^2 - 4x + 7$ $D = 16 - 28 = -12 < 0$ i $a > 1$, then the parabola, which is its graph, is completely located in the upper half-plane, that is $x^2 - 4x + 7 > 0$, and therefore $|x^2 - 4x + 7| = x^2 - 4x + 7$ and equation (1) has the form $x^2 - 4x + 7 + 2 \cdot |x - 2| - 6 = 0$, $x^2 - 4x + 2 \cdot |x - 2| - 1 = 0 | + 3$,

$x^2 - 4x + 4 + 2 \cdot |x - 2| = 3$, $(x - 2)^2 + 2 \cdot (x - 2) - 3 = 0$.

By module properties $(x - 2)^2 = |x - 2|^2$. $|x - 2|^2 + 2|x - 2| - 3 = 0$.

Substitution: $|x - 2| = y$, $y^2 + 2y - 3 = 0$, $y_1 = -3$, $y_2 = 1$. $|x - 2| = -3$, $x \in \varnothing$.

$|x - 2| = 1$, $\left[\begin{array}{l} x - 2 = -1, \\ x - 2 = 1. \end{array}\right.$ $\left[\begin{array}{l} x = 1, \\ x = 3. \end{array}\right.$

Answer: 1; 3.

$|x^2 - 5x + 6| + |x^2 + x - 12| = 0$.

Solution

This equality is possible only when $|x^2 - 5x + 6| = 0$ i $|x^2 + x - 12| = 0$.

To find this value of x, we solve the following system of equations:

$\begin{cases} x^2 - 5x + 6 = 0, \\ x^2 + x - 12 = 0 \end{cases} \Rightarrow -6x + 18 = 0, \ -6x = -18, \ x = \dfrac{-18}{-6}, \ x = 3.$

Answer: 3.

Equations of higher degrees with modules

$|x|^3 + 4x^2 + |x| - 6 = 0$.

Solution:

Replacement $|x| = t$. $t^3 + 4t^2 + t - 6 = 0$.

It is not hard to guess that $t = 1 -$ root of the equation.

$$
\begin{array}{r|l}
t^3 + 4t^2 + t - 6 & t - 1 \\ \cline{2-2}
-\ t^3 - t^2 \quad\quad\quad & t^2 + 5t + 6 \\ \cline{1-1}
5t^2 + t \quad\ \ & \\
-\ 5t^2 - 5t \quad\ \ & \\ \cline{1-1}
6t - 6 & \\
-\ 6t - 6 & \\ \cline{1-1}
0 &
\end{array}
$$

$t^2 + 5t + 6 = 0$. By Vieta's theorem: $t_2 = -2$, $t_3 = -3$.

$\left[\begin{array}{l} |x| = 1, \\ |x| = -2, \\ |x| = -3. \end{array}\right.$ $\left[\begin{array}{l} x_1 = -1, \ x_2 = 1. \\ x \in \varnothing, \\ x \in \varnothing \end{array}\right.$

Answer: -1; 1.

$x^4+4|x|^3-7x^2-22|x|+24=0.$

Solution:

Substitution: $|x|=t.$

$t^4+4t^3-7t^2-22t+24=0.$

Free member divisors: $\pm1;\ \pm2;\ \pm3;\ \pm4;\ \pm6;\ \pm8;\ \pm12;\ \pm24.$

Check the number 1:

$1^4+4\cdot1^3-7\cdot1^2-22\cdot1+24=1+4-7-22+24=0.$

1 – one of the roots of the equation.

$$\begin{array}{l|l}
t^4+4t^3-7t^2-22t+24 & t-1 \\ \cline{2-2}
-\underline{t^4-t^3} & t^3+5t^2-2t-24 \\
\quad 5t^3-7t^2 & \\
\quad -\underline{5t^3-5t^2} & \\
\qquad -2t^2-22t & \\
\qquad \underline{-2t^2+2t} & \\
\qquad\qquad 0 &
\end{array}$$

Check the number 2:

$2^3+5\cdot2^2-2\cdot2-24=28-28=0.$ $t=2-$ second root of the equation.

$$\begin{array}{l|l}
t^3+5t^2-2t-24 & t-2 \\ \cline{2-2}
-\underline{t^3-2t^2} & t^2+7t+12 \\
\quad 7t^2-2t & \\
\quad -\underline{7t^2-14t} & \\
\qquad 12t-24 & \\
\qquad -\underline{12t-24} & \\
\qquad\quad 0 &
\end{array}$$

$t^2+7t+12=0,\ t_3=-3;\ t_4=-4.$

$$\left[\begin{array}{ll}
|x|=1, & x_1=-1;\ x_2=1; \\
|x|=2, & x_3=-2,\ x_4=2; \\
|x|=-3, & x\in\varnothing; \\
|x|=-4 & x\in\varnothing.
\end{array}\right.$$

Answer: $-1;\ 1;\ -2;\ 2.$

Irrational equations with modules

$\sqrt{x^2-2x+1}+\sqrt{x^2+2x+1}=\sqrt{x^2-4x-4}.$

Solution:

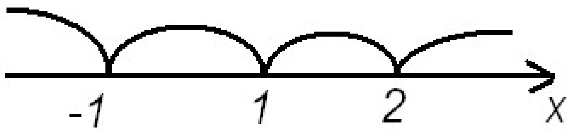
Each of the radical expressions can be represented as a square binomial:

$\sqrt{(x-1)^2}-\sqrt{(x+1)^2}=\sqrt{(x-2)^2}.$

Using the identity $\sqrt{a^2}=|a|,$ we obtain an equation with moduli:

$|x-1|-|x+1|-|x-2|=0,\ \ |x-1|-|x+2|=|x-2.|$

$$\left[\begin{array}{l} x-1=0, \\ x+1=0, \\ x-2=0. \end{array}\right. \quad \left[\begin{array}{l} x=1, \\ x=-1, \\ x=2. \end{array}\right.$$



$$\text{On } (-\infty;\ -1) \quad \begin{array}{ll} x-1<0, & |x-1|=1-x, \\ x+1<0, & |x+1|=-x-1, \\ x-2<0, & |x-2|=2-x. \end{array}$$

$1-x-(-x-1)-(2-x)=0,\ 1-x+x+1-2+x=0,\ \ x=0\notin(-\infty;\ -1).\ \ x\in\varnothing.$

$$\text{On } [-1;\ 1) \quad \begin{array}{ll} x-1<0, & |x-1|=1-x, \\ x+1>0, & |x+1|=x+1, \\ x-2<0, & |x-2|=2-x. \end{array}$$

$1-x-(x+1)-(2-x)=0,\ 1-x-x-1-2+x=0,\ \ -x=2,\ \ x=-2\notin[-1;\ 1);\ \ x\in\varnothing.$

$$\text{On } [1;\ 2) \quad \begin{array}{ll} x-1>0, & |x-1|=1-x, \\ x+1>0, & |x+1|=x+1, \\ x-2<0, & |x-2|=2-x. \end{array}$$

$x-1-x-x-2+x=0,\ \ \ x=4\notin[1;\ 2).\ \ x\in\varnothing.$

$$\text{On } [2;\ +\infty) \quad \begin{array}{ll} x-1>0, & |x-1|=x-1, \\ x+1>0, & |x+1|=x+1, \\ x-2>0, & |x-2|=x-2. \end{array}$$

$x-1-x-1-x+2=0,\ \ \ x=0\notin[2;\ +\infty).\ \ x\in\varnothing.$

Answer: $\varnothing$.

$2\sqrt{x+\sqrt{2x-1}}+\sqrt{x+4-3\sqrt{2x-1}}=\sqrt{32}.$

Solution:

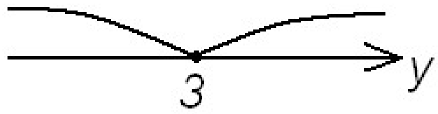
Let's perform the substitution $\sqrt{2x-1}=y,\ \ y\geq 0.\ \left(\sqrt{2x-1}\right)^2=y^2,\ \ 2x-1=y^2,$

$2x=y^2+1,\ \ x=\dfrac{y^2+1}{2}.$ Тогда $\ 2\sqrt{\dfrac{y^2+1}{2}+y}+\sqrt{\dfrac{y^2+1}{2}+4-3y}=\sqrt{32},$

$2\dfrac{\sqrt{y^2+2y+1}}{\sqrt{2}}+\dfrac{\sqrt{y^2-6y+9}}{\sqrt{2}}=4\sqrt{2}\Big|\cdot\sqrt{2};\ \ \ 2\sqrt{(y+1)^2}+\sqrt{(y-3)^2}=8;\ \ \ 2|y+1|+|y-3|=8.$

Since $y\geq 0,$ то $\ y+1\geq 0.$ Then the equation takes the form $2(y+1)+|y-3|=8.$

$y-3=0,\ \ y=3.$



On $(-\infty;\ 3)$ $\ y-3<0,\ \ |y-3|=3-y.\ \ 2y+2+3-y=8,\ \ y=3.\ \ x=\frac{3^2+1}{2}=\frac{9+1}{2}=\frac{10}{2}=5.$

Answer: 5.

$$\frac{\sqrt[3]{|x|+2}}{2}+\frac{\sqrt[3]{|x|+2}}{|x|}=\sqrt[3]{|x|}.$$

Solution:

Replacement $|x|=t,\ \ t>0.$

$\frac{\sqrt[3]{t+2}}{2}+\frac{\sqrt[3]{t+2}}{t}=\sqrt[3]{t};\ \ \sqrt[3]{t+2}\cdot\left(\frac{1}{2}+\frac{1}{t}\right)=\sqrt[3]{t},\ \ \sqrt[3]{t+2}\cdot\frac{t+2}{2t}=\sqrt[3]{t}.$

When raising both sides of the equation to a cube, we get:

$t+2\cdot\frac{(t+2)^3}{(2t)^3}=t;\ \ \frac{(t+2)^4}{8t^3}=t;\ \ (t+2)^4=8t^4.$ We extract the root of the 4th degree:

$$t+2=\pm t\sqrt[4]{8},\quad \left[\begin{array}{l} t+2=-t\sqrt[4]{8}, \\ t+2=t\sqrt[4]{8}, \end{array}\right. \quad \left[\begin{array}{l} t+t\sqrt[4]{8}=-2, \\ t-t\sqrt[4]{8}=-2. \end{array}\right. \quad \left[\begin{array}{l} t\cdot\left(1+\sqrt[4]{8}\right)=-2, \\ t\cdot\left(1-\sqrt[4]{8}\right)=-2, \end{array}\right. \quad \left[\begin{array}{l} t_1=-\frac{2}{1+\sqrt[4]{8}}, \\ t_2=\frac{2}{1-\sqrt[4]{8}}, \end{array}\right.$$

$\left[\begin{array}{l} t_1=-\frac{2}{1+\sqrt[4]{8}}<0, \\ t_2=\frac{2}{1-\sqrt[4]{8}}>0. \end{array}\right.$ $\ t_1-$ does not satisfy the condition $t>0.$ Means $|x|=\frac{2}{\sqrt[4]{8}-1};$

$x_1=-\frac{2}{\sqrt[4]{8}-1};\quad x_2=\frac{2}{\sqrt[4]{8}-1}.$

Answer: $x_1=-\frac{2}{\sqrt[4]{8}-1};\quad x_2=\frac{2}{\sqrt[4]{8}-1}.$

$$2x^2-2x-6+\sqrt{|6+x-x^2|}=0.$$

Solution:

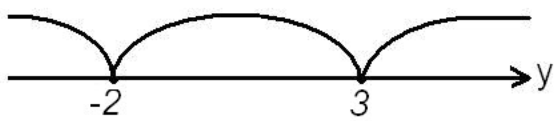
$|6+x-x^2|=|-(x^2-x-6)|=|x^2-x-6|$ (module property).

Then the equation will have the form:

$2x^2-2x-6+\sqrt{|x^2-x-6|}=0.\ \ x^2-x-6=0;\ \ x_1=-2;\ \ x_2=3.$

$x^2-x-6=(x+2)\cdot(x-3).\ \ 2x^2-2x-6+\sqrt{|(x+2)\cdot(x-3)|}=0.$

Consider this equation in the intervals:



On $(-\infty;-2)$: $\begin{matrix} x+2<0, \\ x-3<0. \end{matrix}$ $(x+2)(x-3)>0.$ $|(x+2)(x-3)|=(x+2)(x-3).$

$2x^2-2x-6+\sqrt{x^2-x-6}=0;$ $2(x^2-x)-6+\sqrt{x^2-x-6}=0.$

We denote $\sqrt{x^2-x-6}=t,$ then $x^2-x-6=t^2,$ $x^2-x=t^2+6.$ $2\cdot(t^2+6)-6+t=0,$

$2t^2+12-6+t=0,$ $2t^2+t+6=0,$ $D=1-4\cdot 2\cdot 6=1-45=-47<0,$ $t\in\varnothing.$

On $[-2;3)$: $\begin{matrix} x+2>0, \\ x-3<0. \end{matrix}$ $(x+2)(x-3)<0.$ $|x^2-x-6|=6+x-x^2,$

$2(x^2-x)-6+\sqrt{6+x-x^2}=0.$

We denote $\sqrt{6+x-x^2}=y,$ $y\geq 0,$ $-x^2+x+6=y^2,|\times(-1),$ $x^2-x-6=-y^2,$ $x^2-x=6-y^2,$

$2(6-y^2)-6+y=0,$ $12-2y^2-6+y=0,$ $-2y^2+y+6=0|\times(-1),$ $2y^2-y-6=0.$

$D=1+8\cdot 6=49,$ $y_1=\frac{1-7}{4}=-\frac{6}{4}=-\frac{3}{2};$ $y_2=\frac{1+7}{4}=2.$

$-\frac{3}{2}-$ does not satisfy the condition $y\geq 0.$

Means, $\sqrt{6+x-x^2}=2,$ $6+x-x^2=4,$ $-x^2+x+2=0,$ $x^2-x-2=0.$ $x_1=-1,$

$x_2=2\in[-2;\ 3.)$

On $[3;\ +\infty)$: $\begin{matrix} x+2>0, \\ x-3>0. \end{matrix}$ $(x+2)(x-3)>0.$ $|(x+2)(x-3)|=(x+2)(x-3),$

$2(x^2-x)-6+\sqrt{x^2-x-6}=0,$ $\sqrt{x^2-x-6}=V;$ $V\geq 0.$ $x^2-x-6=V^2;$ $x^2-x=V^2+6;$

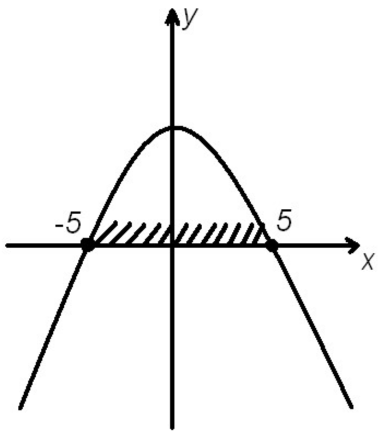
$2(V^2+6)-6+V=0,$ $2V^2+V+6=0,$ $D=1-48=-47<0,$ $V\in\varnothing,$ $x\in\varnothing.$

Answer: $-1;\ 2.$

$\sqrt[3]{|x+5|}-\sqrt[3]{|x-5|}=\sqrt[6]{25-x^2}.$

Solution:

We find the range of permissible values: $25-x^2\geq 0,$ $(5-x)\cdot(5+x)\geq 0.$ $x\in[-5;\ 5].$ On this interval $|x+5|=x+5;$ $|x-5|=-(x-5).$



Then this equation takes the form:

$\sqrt[3]{x+5}-\sqrt[3]{-(x-5)}=\sqrt[6]{25-x^2};$ $\sqrt[3]{x+5}+\sqrt[3]{x-5}=\sqrt[6]{25-x^2};$

Let's raise both parts to the third degree:

$x+5+3\left(\sqrt[3]{x+5}\right)^2\cdot\sqrt[3]{x-5}+3\sqrt[3]{x+5}\cdot\left(\sqrt[3]{x-5}\right)^2+x-5=\sqrt{25-x^2};$

$2x+\sqrt[3]{x+5}\cdot\sqrt[3]{x-5}\cdot\left(\sqrt[3]{x+5}+\sqrt[3]{x-5}\right)=\sqrt{25-x^2};$

$2x+3\sqrt[3]{x^2-25}\cdot\sqrt[6]{25-x^2}=\sqrt{25-x^2};\;\; 2x-3\sqrt[3]{25-x^2}\cdot\sqrt[6]{25-x^2}=\sqrt{25-x^2};$

$2x-3\sqrt[3]{\left(25-x^2\right)^2}\cdot\sqrt[6]{25-x^2}=\sqrt{25-x^2};\;\; 2x-3\sqrt[6]{\left(25-x^2\right)^3}=\sqrt{25-x^2};$

$2x-3\sqrt{25-x^2}=\sqrt{25-x^2};\;\; 2x=\sqrt{25-x^2}+3\sqrt{25-x^2};\;\; 2x=4\sqrt{25-x^2}\,\big|:2;\;\; x=2\sqrt{25-x^2},$

$x>0.\;\; x^2=4\left(25-x^2\right),\;\; x^2=100-4x^2,\;\; 5x^2=100,\;\; x^2=\frac{100}{5};\;\; x^2=20;\;\; x=\pm\sqrt{20}=\pm2\sqrt{5};$

$-2\sqrt{5}-$ does not satisfy the condition $x>0.$

Answer: $2\sqrt{5}.$

Exponential and logarithmic equations with modules

$2^{\left|x^2-3x+5\right|}=32.$

Solution:

$2^{\left|x^2-3x+5\right|}=2^5.$ By virtue of the identities of the exponential function, we have: $\left|x^2-3x+5\right|=5.$ This equation is equivalent to the set of equations:

$$\left[\begin{array}{l} x^2-3x+5=-5, \\ x^2-3x+5=5. \end{array}\right. \quad \left[\begin{array}{l} x^2-3x+10=0,\;\; D=9-40=-31<0,\;\; x\in\varnothing. \\ x^2-3x=0. \qquad x(x-3)=0,\; \left[\begin{array}{l} x_1=0, \\ x_2=3. \end{array}\right. \end{array}\right.$$

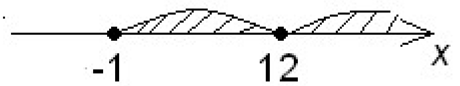
Answer: 0; 3.

$16\cdot\sqrt{\left(\frac{1}{4}\right)^{\left|3-\frac{x}{4}\right|}}=2^{\sqrt{x+1}}.$

Solution:

$2^4\cdot\sqrt{\left(2^{-2}\right)^{\left|3-\frac{x}{4}\right|}}=2^{\sqrt{x+1}};\;\; 2^4\cdot\sqrt{2^{\left|-6+\frac{x}{4}\right|}}=2^{\sqrt{x+1}};\;\; 2^4\cdot\sqrt{2^{-\frac{|x-12|}{2}}}=2^{\sqrt{x+1}};\;\; 2^4\cdot\left(2^{-\frac{|x-12|}{2}}\right)^{\frac{1}{2}}=2^{\sqrt{x+1}};$

$2^4\cdot2^{-\frac{|x-12|}{4}}=2^{\sqrt{x+1}};\;\; 2^{4+|x-12|}=2^{\sqrt{x+1}};\;\; 4-\frac{|x-12|}{4}=\sqrt{x+1};\qquad \begin{array}{ll} x-12=0, & x+1\geq0, \\ x=12. & x\geq-1. \end{array}$



On $[-1;\ 12)$: $x-12<0,\ |x-12|=2-x,$

$4-\frac{12-x}{4}=\sqrt{x+1}\,\big|\times4,\;\; 16-12+x=4\sqrt{x+1},\;\; 4+x=4\sqrt{x+1},\;\; 16+8x+x^2=16x+16,$

$x^2-8x=0,\;\; x(x-8)=0,\;\; x_1=0,\;\; x_2=8\in[-1;\ 12).$

On $[12;\ +\infty)$: $x-12>0,\ |x-12|=x-12.$

$4-\frac{x-12}{4}=\sqrt{x+1}\Big|\times 4,\ 16-x+12=4\sqrt{x+1},\ 28-x=4\sqrt{x+1},\ x^2-72x+708=0,$

$D=5184-3072=2112=64\cdot 33.$

$x_1=\frac{72-8\sqrt{33}}{2}=36-4\sqrt{33},\ x_2=36+4\sqrt{33}$ – *не задовольняє перевірку*.

Answer: 0; 8; $36-4\sqrt{33}.$

$\left(\frac{1}{2}\left(\sqrt{14}-\sqrt{2}\right)\right)^{|x|}+\left(\sqrt{4-\sqrt{7}}\right)^{|x|}=2\cdot\left(4-\sqrt{7}\right).$

Solution:

$\sqrt{4-\sqrt{7}}=\sqrt{\frac{\left(4-\sqrt{7}\right)\cdot 4}{4}}=\sqrt{\frac{16-4\sqrt{7}}{4}}=\sqrt{\frac{14-2\cdot\sqrt{7}\cdot 2\sqrt{2}+2}{4}}=$

$\sqrt{\frac{\left(\sqrt{14}-\sqrt{2}\right)^2}{2^2}}=\left|\frac{\sqrt{14}-\sqrt{2}}{2}\right|=\frac{\sqrt{14}-\sqrt{2}}{2}\to\sqrt{14}-\sqrt{2}=2\sqrt{4-\sqrt{7}};$

$\left(\frac{1}{2}\cdot 2\sqrt{4-\sqrt{7}}\right)^{|x|}+\left(\sqrt{4-\sqrt{7}}\right)^{|x|}=2\cdot\left(4-\sqrt{7}\right);$

$\left(\sqrt{4-\sqrt{7}}\right)^{|x|}+\left(\sqrt{4-\sqrt{7}}\right)^{|x|}=2\cdot\left(4-\sqrt{7}\right);$

$2\cdot\left(\sqrt{4-\sqrt{7}}\right)^{|x|}=2\cdot\left(4-\sqrt{7}\right):2,$

$\left(\sqrt{4-\sqrt{7}}\right)^{|x|}=4-\sqrt{7};\ \left(4-\sqrt{7}\right)^{\frac{|x|}{2}}=4-\sqrt{7};$

$\frac{|x|}{2}=1;\ |x|=2;\ x_1=-2;\ x_2=2.$

Answer: $x_1=-2;\ x_2=2.$

$8\cdot 3^{|x+1|}-3^{2\cdot|x+1|}+9=0.$

Solution:

Replacement $3^{|x+1|}=t,$ leads to the equation:

$8t-t^2+9=0\big|\cdot(-1),\ t^2-8t-9=0.$ By Vieta's theorem $t_1=9,\ t_2=-1$ does not satisfy the condition $t>0.$

$3^{|x+1|}=9,\ 3^{|x+1|}=3^2,\ |x+1|=2\ \left[\begin{array}{l}x+1=-2,\\ x+1=2.\end{array}\right.\ \left[\begin{array}{l}x=-3,\\ x=1.\end{array}\right.$

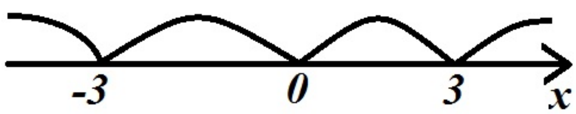
Answer: $-3;1.$

$2^{|x-3|}+2^{|x+3|}-4\cdot 2^{|x|}-132=0.$

Solution:

The left side of this equation is an even function. This means that its roots are opposite in pairs.

$\left[\begin{array}{l}x-3=0,\\ x+3=0,\\ x=0\end{array}\right.\ \left[\begin{array}{l}x=3,\\ x=-3,\\ x=0\end{array}\right.$



Let's define the integral roots.

They are at integral intervals:

$[0;3)$: $x-3<0,\ |x-3|=-(x-3),\ x+3>0,\ |x+3|=x+3,\ |x|=x$

Then the equation takes the form: $2^{3-x}+2^{x+3}-4\cdot 2^x-132=0.$

Analysis of the left side of the equation shows that $2^{3-x}<8,\ 2^{x+3}<64,\ 4\cdot 2^x>4.$

$8+64-4\neq 132,$ то есть на $[0;\ 3)$ $x\in\varnothing.$

On $[3;\ +\infty)$ $x-3>0,\ |x-3|=x-3,$ $\begin{matrix} x+3>0, & |x+3|=x+3, \\ x>0 & |x|=x. \end{matrix}$

The equation has the form:

$2^{x-3}+2^{x+3}-4\cdot 2^x-132=0,$

$2^x\cdot\frac{1}{2^3}+2^x\cdot 2^3-4\cdot 2^{x\cdot}-132=0,$

$2^x\cdot\left(\frac{1}{8}+8-4\right)-132=0,\ \ 2^x\cdot 4\frac{1}{8}=132,\ \ 2^x=132:\frac{33}{8},\ \ 2^x=\frac{132}{1}\cdot\frac{8}{33}=32,\ \ 2^x=2^5,\ \ x=5.$

The opposite number $-5-$ also root.

Answer: $-5;\ 5.$

$$\left(\sqrt{5-2\sqrt{6}}\right)^{|x^2+x-12|}+\left(\sqrt{5+2\sqrt{6}}\right)^{|x^2+x-12|}=10.$$

Solution:

Consider the product of conjugate numbers:

$\left(5-2\sqrt{6}\right)\cdot\left(5+2\sqrt{6}\right)=25-4\cdot 6=1.$ From here $5+2\sqrt{6}=\frac{1}{5-2\sqrt{6}}.$

Let's introduce a new variable: $\left(\sqrt{5-2\sqrt{6}}\right)^{|x^2+x-12|}=y,$ then $\left(\sqrt{5+2\sqrt{6}}\right)^{|x^2+x-12|}=\frac{1}{y}$ и, substituting into this equation, we obtain:

$y+\frac{1}{y}=10\Big|\cdot y,\ \ y^2-10y+1=0,\ \ D=100-4=96.$

$y_1=\frac{10-\sqrt{96}}{2}=\frac{10-\sqrt{16\cdot 6}}{2}=\frac{10-4\sqrt{6}}{2}=\frac{2\cdot\left(5-2\sqrt{6}\right)}{2}=5-2\sqrt{6};$

$y_2=5+2\sqrt{6}.$

Taking into account the change made, we obtain the following set of equations:

$$\left[\begin{array}{l}\left(\sqrt{5-2\sqrt{6}}\right)^{|x^2+x-12|}=5-2\sqrt{6},\\ \left(\sqrt{5-2\sqrt{6}}\right)^{|x^2+x-12|}=5+2\sqrt{6}.\end{array}\right.\quad \left[\begin{array}{l}\left(5-2\sqrt{6}\right)^{\frac{|x^2+x-12|}{2}}=5-2\sqrt{6},\ \ \frac{|x^2+x-12|}{2}=1,\ \ |x^2+x-12|=2,\\ \left(5-2\sqrt{6}\right)^{\frac{|x^2+x-12|}{2}}=5+2\sqrt{6}.\end{array}\right.$$

$$\left[\begin{array}{l}x^2+x-12=-2,\\ x^2+x-12=2.\end{array}\right.\quad \left[\begin{array}{l}x^2+x-10=0,\ \ D=1+40=41,\\ x^2+x-14=0.\ \ D=1+56=57\end{array}\right.\quad \begin{array}{l}x_1=\frac{-1-\sqrt{41}}{2},\ \ x_2=\frac{-1+\sqrt{41}}{2},\\ x_3=\frac{-1-\sqrt{57}}{2},\ \ x_4=\frac{-1+\sqrt{57}}{2}.\end{array}$$

Because $5+2\sqrt{6}=\frac{1}{5-2\sqrt{6}}=\left(5-2\sqrt{6}\right)^{-1},$ then

$\left(5-2\sqrt{6}\right)^{\frac{|x^2+x-12|}{2}}=\left(5-2\sqrt{6}\right)^{-1};\ \ \frac{|x^2+x-12|}{2}=-1;\ \ |x^2+x-12|=-2.$

By the properties of the modulus, this equation has no roots.

Answer: $\frac{-1-\sqrt{41}}{2}$; $\frac{-1+\sqrt{41}}{2}$; $\frac{-1-\sqrt{57}}{2}$; $\frac{-1+\sqrt{57}}{2}$.

$16^{\sqrt{|x-1|}}+12^{\sqrt{|x-1|}}=20^{\sqrt{|x-1|}}$.

Solution:

Introducing a new variable $\sqrt{|x-1|}=t$, then $16^t+12^t=20^t \Big| : 20^t$; $\left(\frac{16}{20}\right)^t+\left(\frac{12}{20}\right)^t=1$,

$\left(\frac{4}{5}\right)^t+\left(\frac{3}{5}\right)^t=1$.

By selection we make sure that $t=2-$ the root of this equation. Really,

$\left(\frac{4}{5}\right)^2+\left(\frac{3}{5}\right)^2=\frac{16}{25}+\frac{9}{25}=\frac{16+9}{25}=\frac{25}{25}=1$. Since the functions $y=\left(\frac{4}{5}\right)^t$ i $y=\left(\frac{3}{5}\right)^t$ – decreasing, then the function $y=\left(\frac{4}{5}\right)^t+\left(\frac{3}{5}\right)^t$ – decreasing, which means it has only one root $y=2$.

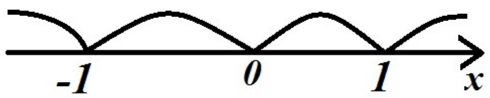
Returning to the substitution, we get $\sqrt{|x-1|}=2$, $\left(\sqrt{|x-1|}\right)^2=2^2$, $|x-1|=4$.

$\left[\begin{array}{l} x-1=-4, \\ x-1=4 \end{array}\right.$ $\left[\begin{array}{l} x=-3, \\ x=5. \end{array}\right.$

Answer: -3; 5.

$2^{-|x|}=\frac{1}{2\sqrt{2}}\cdot\left(|x+1|+|x-1|\right)$.

Solution:

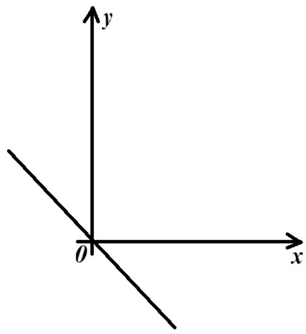


On $(-\infty;\,-1)$ $2^x=-\frac{x}{\sqrt{2}}\Big|\cdot\sqrt{2}$, $2^x\cdot 2^{\frac{1}{2}}=-x$, $2^{x+\frac{1}{2}}=-x$, $x\in\varnothing$

On $[-1;\,0)$ $2^x=\frac{1}{\sqrt{2}}$; $2^{x+\frac{1}{2}}=2^0$; $x=-\frac{1}{2}$;

On $[0;\,1)$ $2^{-x}=\frac{1}{\sqrt{2}}$; $2^{\frac{1}{2}-x}=1$; $x=\frac{1}{2}$;

On $[1;\,+\infty)$ $2^{-x}=\frac{x}{\sqrt{2}}$; $2^{\frac{1}{2}-x}=x$; $x\in\varnothing$.



Answer: $-\frac{1}{2}$; $\frac{1}{2}$.

$|\lg x|=3$.

Solution:

Range of valid values: $(0;\,+\infty)$.

$\left[\begin{array}{l} \lg x=-3, \\ \lg x=3. \end{array}\right.$ By the definition of the logarithm of a number, we have:

$$\left[\begin{array}{l} x=10^{-3}, \\ x=10^{3} \end{array}\right. \left[\begin{array}{l} x=0{,}001, \\ x=1000. \end{array}\right.$$

Answer: 0,001; 1000.

$\log_3 \log_2 |x| = 0.$

Solution:

By the definition of the logarithm of a number, we have: $\log_2 |x| = 3^0;$

$$\log_2 |x| = 1,\ \ |x| = 2^1;\ \ |x| = 2,\ \left[\begin{array}{l} x=-2, \\ x=2. \end{array}\right.$$

Answer: $-2;\ 2.$

$\log_3 \left|\log_2 |x|\right| = 0.$

Solution:

$$\left|\log_2 |x|\right| = 3^0;\ \ \left|\log_2 |x|\right| = 1,\ \left[\begin{array}{l} \log_2 |x| = -1, \\ \log_2 |x| = 1 \end{array}\right. \left[\begin{array}{l} |x| = 2^{-1}, \\ |x| = 2 \end{array}\right. \left[\begin{array}{l} |x| = \dfrac{1}{2}, \\ |x| = 2 \end{array}\right.$$

$$x_1 = \frac{1}{2};\ \ x_2 = -\frac{1}{2};\ \ x_3 = 2;\ \ x_4 = -2.$$

Answer: $\frac{1}{2};\ -\frac{1}{2};\ 2;\ -2.$

$2\log_{|x|} 25 - 3\log_{25}|x| = 1.$

Solution:

Let's move on to the base of logarithms 25:

$$\log_{|x|} 25 = \frac{\log_{25} 25}{\log_{25}|x|} = \frac{1}{\log_{25}|x|}.$$

This equation takes the form:

$$\frac{2}{\log|x|} - 3\log_{25}|x| = 1\Big|\cdot \log_{25}|x|,\ \ 2 - 3\left(\log_{25}|x|\right) = \log_{25}|x|,\ \ 3\left(\log_{25}|x|\right)^2 + \log_{25}|x| - 2 = 0.$$

We denote $\log_{25}|x| = y,$ then

$$3y^2 + y - 2 = 0,\ \ D = 1 + 24 = 25.\ \ y_1 = \frac{-1-5}{6} = -1;\ \ y_2 = \frac{-1+5}{6} = \frac{2}{3};$$

$$\log|x| = -1,\ \ |x| = 25^{-1};\ \ |x| = \frac{1}{25};\ \left[\begin{array}{l} x = -\dfrac{1}{25}, \\ x = \dfrac{1}{25}. \end{array}\right.$$

$$\log_{25}|x| = \frac{2}{3};\ \ |x| = 25^{\frac{2}{3}};\ \ |x| = \sqrt[3]{625},\ \left|\begin{array}{l} x = -\sqrt[3]{625}, \\ x = \sqrt[3]{625}. \end{array}\right.$$

Answer: $-\frac{1}{25};\ \frac{1}{25};\ -\sqrt[3]{625};\ \sqrt[3]{625}.$

$$\frac{|2\lg x|}{|\lg(5x-4)|} = 1.$$

Solution:

According to the properties of the modulus of a number, this equation can be represented as:

$$\left|\frac{2\lg x}{\lg(5x-4)}\right|=1, \qquad \left[\begin{array}{l}\dfrac{2\lg x}{\lg(5x-4)}=-1,\\ \dfrac{2\lg x}{\lg(5x-4)}.\end{array}\right. \quad \left[\begin{array}{l}2\lg x=-\lg(5x-4),\\ 2\lg x=\lg(5x-4).\end{array}\right. \quad \left[\begin{array}{l}\lg x^2=\lg(5x-4)^{-1},\\ \lg x^2=\lg(5x-4)\end{array}\right.$$

$$ОДЗ:\begin{cases}x>0\\ 5x-4>0,\ x>\dfrac{4}{5};\end{cases}$$

$$\left[\begin{array}{l}x^2=(5x-4)^{-1},\\ x^2=5x-4.\end{array}\right. \quad \left[\begin{array}{l}x^2=\dfrac{1}{5x-4,}\\ x^2-5x+4=0.\end{array}\right.$$

$\left[\begin{array}{l}5x^3-4x^2-1=0,\\ x_1=1;\ x_2=4.\end{array}\right.$ It is easy to see that 1 is the root of this equation.

Then

$$\begin{array}{l|l} 5x^3-4x^2-1 & x-1 \\ \hline \underline{5x^3-5x^2} & 5x^2+x+1 \\ \quad x^2-1 & \\ \quad \underline{x^2-x} & \\ \qquad x-1 & \\ \qquad \underline{x-1} & \\ \qquad\ \ 0 & \end{array}$$

$5x^2+x+1=0,\ D=1-20=-19<0$

$x\in\varnothing.$

At $x=1$ the left side of the equation does not exist, although 1 is included in the range of admissible values.

Answer: 4.

$$\log_2|2x-5|+\log_2|x+20|=\frac{1}{\lg 2}.$$

Solution:

Let's move on to the base of logarithms 2:

$$\frac{1}{\lg 2}=\frac{1}{\dfrac{\log_2 2}{\log_2 10}}=\frac{1}{\dfrac{1}{\log_2 10}}=\log_2 10.$$

The original equation takes the form:

$$\log_2|2x-5|+\log_2|x+20|=\log_2 10.$$

We use the theorem on the sum of logarithms:

$$\log_2(|2x-5|+|x+20|)=\log_2 10.$$

Since the logarithmic function is monotonic, we have: $|2x-5|\cdot|x+20|=10.$

By the properties of the modulus of a number, we have:

$|(2x-5)\cdot(x+20)|=10,$ $\left[\begin{array}{l}(2x-5)(x+20)=-10,\\(2x-5)(x+20)=10.\end{array}\right.$ $\left[\begin{array}{l}2x^2+40x-5x-100=-10,\\2x^2+40x-5x-100=10.\end{array}\right.$

$\left[\begin{array}{l}2x^2+35x-90=0,\\2x^2+35x-110=0.\end{array}\right.$ $D=1225+720=1945.$ $x_1=\dfrac{-35-\sqrt{1945}}{4};$ $x_2=\dfrac{-35\sqrt{1945}}{4};$

$D=1225+880=2105.$ $x_3=-35-\sqrt{2105};$ $x_4=-35+\sqrt{2105}.$

Answer: $\dfrac{-35-\sqrt{1945}}{4};$ $\dfrac{-35\sqrt{1945}}{4};$ $-35-\sqrt{2105};$ $-35+\sqrt{2105}.$

$\log_{|x+1|}\left|2x^2-3x+1\right|=2.$

Solution:

By the definition of the logarithm of a number, we have:

$|x+1|^2=\left|2x^2-3x+1\right|.$ Considering the fact that $|x+1|^2=(x+1)^2$ we obtain the equation: $(x+1)^2=\left|2x^2-3x+1\right|.$ It is equivalent to such a collection:

$\left[\begin{array}{l}2x^2-3x+1=-(x+1)^2,\\2x^2-3x+1=(x+1)^2.\end{array}\right.$ $\left[\begin{array}{l}2x^2-3x+1=-x^2-2x-1,\\2x^2-3x+1=x^2+2x+1.\end{array}\right.$

$\left[\begin{array}{l}3x^2-x+2=0,\ D=1-24=-23<0,\ x\in\varnothing.\\x^2-5x=0\quad \left[\begin{array}{l}x=0,\\x=5.\end{array}\right.\end{array}\right.$

At $x=0$ expression $|x+1|=1,$ and the left side of the equation expresses the content.

Answer: 5.

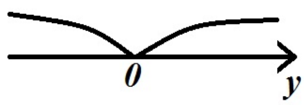
$\log_3|x|+\log_{|x|}81-\left|\dfrac{\log_5|x|}{\log_5 3}\right|=2.$

Solution:

$\dfrac{\log_5|x|}{\log_5 3}=\dfrac{\dfrac{\log_3|x|}{\log_3 5}}{\dfrac{\log_3 3}{\log_3 5}}=\dfrac{\log_3|x|}{1}=\log_3|x|.$ Then the equation takes the form:

$\log_3|x|+\log_{|x|}81-\left|\log_3|x|\right|=2;$ $\log_{|x|}81=\dfrac{\log_3 81}{\log_3|x|}=\dfrac{4}{\log_3|x|};$ $\log_3|x|+\dfrac{4}{\log_3|x|}-\left|\log_3|x|\right|=2.$

Let's introduce a new variable: $\log_3|x|=y.$ $y+\dfrac{4}{y}-|y|=2,$ $y=0.$



On $(-\infty;\ 0]$ $y<0,$ $|y|=-y;$ $y+\dfrac{4}{y}-(-y)=2,$ $y^2+y^2+4-2y=0,$ $2y^2-2y+4=0,$

$y^2-y+2=0,$ $D=1-8=-7<0,$ $y\in\varnothing.$

On $(0;\ +\infty)$ $y>0,$ $|y|=y;$ $y+\dfrac{4}{y}-y=2,$ $\dfrac{y}{2}=2,$ $y=2.$ $\log_3|x|=2,$ $|x|=3^2,$ $|x|=9,$

$x=-9,$ $x=9.$

Answer: $-9;\ 9.$

$\log_{|x|}|x+1|\cdot\log_{|x+1|}|x+2|\cdot\left|\log_{|x+2|}|x+3|\cdot\ldots\cdot\log_{|x+19|}|x+20|\right|=2.$

Solution:

Apply the base transition formula $|x|$:

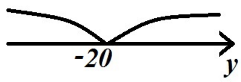
$$\log_{|x+1|}|x+2|=\frac{\log_{|x|}|x+2|}{\log_{|x|}|x+1|},\ \log_{|x+2|}|x+3|=\frac{\log_{|x|}|x+3|}{\log_{|x|}|x+2|},\ \log_{|x+19|}|x+20|=\frac{\log_{|x|}|x+20|}{\log_{|x|}|x+19|},$$

$$\log_{|x|}|x+1|\cdot\frac{\log_{|x|}|x+2|}{\log_{|x|}|x+1|}\cdot\frac{\log_{|x|}|x+3|}{\log_{|x|}|x+2|}\cdot\ldots\cdot\frac{\log_{|x|}|x+20|}{\log_{|x|}|x+19|}=2.$$

Finally, we have the equation: $\log_{|x|}|x+20|=2.$ By the definition of the logarithm

$|x|^2=|x+20|,\ x^2=|x+20|,\ x+20=0,\ x=-20$

On $(-\infty;\ -20)$ $x+20<0,\ |x+20|=-(x+20).$



$x^2=-(x+20);\ x^2+x+20=0,\ \ D<0,\ \ x\in\varnothing.$

On $[-20;\ +\infty)$ $x+20>0,\ |x+20|=x+20.$

$x^2=x+20);\ x^2-x-20=0,\ \ x_1=-4;\ x_2=5$ (Vieta's theorem).

At $x=-4$ $\log_{|x+1|}|x+1|$ loses its meaning.

Answer: 5.

$$\log_8|x|+\log_8^2|x|+\log_8^3|x|+\ldots=\frac{1}{2}.$$
$x>0$

Solution:

The left side of the equation is the sum of an infinite decreasing geometric progression with the denominator $q=\frac{\log_8^2|x|}{\log_8|x|}=\log_8|x|.$

$$S=\frac{b_1}{1-q};\ S=\frac{\log_8|x|}{1-\log_8|x|};\ \frac{\log_8|x|}{1-\log_8|x|}=\frac{1}{2};\ 2\log_8|x|=1-\log_8|x|,\ \ 3\log_8|x|=1,\ \ \log_8|x|=\frac{1}{3};$$

$$|x|=8^{\frac{1}{3}};\ |x|=2,\ \left[\begin{array}{l}x=-2\ \text{не задовольняє умову } x>0\\ x=2.\end{array}\right.$$

Answer: 2.

<u>Trigonometric equations with modules</u>

$$\left|Sin\left(2x-\frac{\pi}{3}\right)\right|=\frac{1}{2}.$$

Solution:

This equation is equivalent to a set of such equations:

$$\left[\begin{array}{l}Sin\left(2x-\frac{\pi}{3}\right)=-\frac{1}{2},\\ Sin\left(2x-\frac{\pi}{3}\right)=\frac{1}{2}\end{array}\right.\ \left[\begin{array}{l}2x-\frac{\pi}{3}=(-1)^n\cdot\arcsin\left(-\frac{1}{2}\right)+\pi n,\\ 2x-\frac{\pi}{3}=(-1)^n\cdot\arcsin\frac{1}{2}+\pi n\end{array}\right.\ \left[\begin{array}{l}2x-\frac{\pi}{3}=(-1)^n\cdot\left(-\frac{\pi}{6}\right)+\pi n,\\ 2x-\frac{\pi}{3}=(-1)^n\cdot\frac{\pi}{6}+\pi n\end{array}\right.$$

$$\left[\begin{array}{l}2x-\frac{\pi}{3}=-(-1)^n\cdot\frac{\pi}{6}+\pi n,\\ 2x-\frac{\pi}{3}=(-1)^n\cdot\frac{\pi}{6}+\pi n\end{array}\right.\ \left[\begin{array}{l}2x=\frac{\pi}{3}-(-1)^n\cdot\frac{\pi}{6}+\pi n,|:2\\ 2x=\frac{\pi}{3}+(-1)^n\cdot\frac{\pi}{6}+\pi n\,|:2\end{array}\right.\ \left[\begin{array}{l}x=\frac{\pi}{6}-(-1)^n\cdot\frac{\pi}{12}+\frac{\pi n}{2},\\ x=\frac{\pi}{6}+(-1)^n\cdot\frac{\pi}{12}+\frac{\pi n}{2}\end{array}\right.$$
либо:

$x=\frac{\pi}{6}\pm\frac{\pi}{12}+\frac{\pi n}{2},\ n\in Z.$

Answer: $\frac{\pi}{6}\pm\frac{\pi}{12}+\frac{\pi n}{2},\ n\in Z.$

$\left|\cos\frac{x}{2}\right|=-0{,}3.$

Solution:

$x\in\varnothing.$

Answer: $\varnothing.$

$\left|tg\left(x-20^{\circ}\right)\right|=2.$

Solution:

$\left[\begin{array}{l} tg\left(x-20^{\circ}\right)=-2, \\ tg\left(x-20^{\circ}\right)=2 \end{array}\right.\ \left[\begin{array}{l} x-20^{\circ}=arctg(-2)+180^{\circ}n, \\ x-20^{\circ}=arctg2+180^{\circ}n \end{array}\right.\ \left[\begin{array}{l} x=20-arctg2+180n, \\ x=20+arctg2+180n \end{array}\right.$

$x=20^{\circ}\pm arctg2+180^{\circ}n,\ n\in Z.$

Answer: $20^{\circ}\pm arctg2+180^{\circ}n.$

$tg\frac{\pi}{|x-1|}=-1.$

Solution:

$\frac{\pi}{|x-1|}=arctg(-1)+\pi n,\ \frac{\pi}{|x-1|}=-\frac{\pi}{4}+\pi n,\ \frac{\pi}{|x-1|}=\frac{-\pi+4\pi n}{4}\Big|:\pi\ \ \frac{1}{|x-1|}=\frac{-1+4n}{4};$

$|x-1|=\frac{4}{4n-1};\ \left[\begin{array}{l} x-1=-\frac{4}{4n-1};\ x=1-\frac{4}{4n-1}; \\ x-1=\frac{4}{4n-1};\ x=1+\frac{4}{4n-1} \end{array}\right.\ n=1,2,3\ldots\ \left[\begin{array}{l} x=\frac{4n-1-4}{4n-1}=\frac{4n-5}{4n-1}; \\ x=\frac{4n+3}{4n-1}. \end{array}\right.$

Answer: $\frac{4n-5}{4n-1},\ \frac{4n+3}{4n-1},\ n\in\{1,2,3\ldots\}.$

$\left|Sin\pi\cdot x^{2}\right|=1.$

Solution:

$\left[\begin{array}{l} Sin\pi\cdot x^{2}=-1, \\ Sin\pi\cdot x^{2}=1 \end{array}\right.$ If the Sine of an angle is $\pm1,$ then the cosine of this angle is 0, that is

$Cos\pi\cdot x^{2}=0,\ \pi x^{2}=\frac{\pi}{2}+k\pi,\ x^{2}=\frac{1}{2}+k.$

This equation has a solution for $k=\{0,1,2,\ldots\}\quad x=\pm\sqrt{\frac{1}{2}+k}.$

Answer: $\pm\sqrt{\frac{1}{2}+k},\ k-$ non-negative integer.

$tg\left|\pi\sin\pi|x|\right|=\sqrt{3}.$

Solution:

$\left|\pi\sin\pi|x|\right|=\frac{\pi}{3}+k\pi\Big|:\pi,\ \left|\sin\pi|x|\right|=\frac{1}{3}+k.$ Because the $0\le\left|\sin\pi|x|\right|\le1,$ then $k=0$ (only).

Then $\left|\sin\pi|x|\right|=\frac{1}{3}.$ Let's square both sides of the equation.

$\sin^2\pi|x|=\frac{1}{9}$. Lowering the degree:

$\frac{1-\cos 2\pi|x|}{2}=\frac{1}{9}\Big|\cdot 18\;\; 9-9\cos 2\pi|x|=2,\;\; -\cos 2\pi|x|=-7,\;\; \cos 2\pi|x|=\frac{7}{9},$

$2\pi|x|=\pm\arccos\frac{7}{9}+2\pi|x|,\;\; |x|=\pm\frac{1}{2\pi}\arccos\frac{7}{9}+\pi,\;\; n\in N$ and $n=0.$

$x=n\pm\arccos\frac{7}{9}$ at $n\in N$.

$x=\pm\frac{1}{2\pi}\arccos\frac{7}{9}$ at $n=0.$

Answer: $n\pm\arccos\frac{7}{9}$ at $n\in N$, $\pm\frac{1}{2\pi}\arccos\frac{7}{9}$ at $n=0.$

$\left|\sin|x|\cdot\cos x\right|\cdot\left|3-4\sin^2 x\right|\cdot\left|4\cos^2 x-3\right|=\frac{1}{2}.$

Solution:

Because the $\cos x=\cos|x|$ and $\sin^2 x=\sin^2|x|$, then the equation takes the form

$\left|\sin|x|\cdot\cos|x|\right|\cdot\left(3-4\sin^2|x|\right)\cdot\left(4\cos^2|x|-3\right)=\frac{1}{2},$

$\left|\sin|x|\cdot\left(3-4\sin^2|x|\right)\cdot\cos|x|\cdot\left(4\cos^2|x|-3\right)\right|=\frac{1}{2},$

$\left|\left(3\sin|x|-4\sin^3|x|\right)\cdot\left(4\cos^3|x|-3\cos|x|\right)\right|=\frac{1}{2}.$

$\left|\sin|3x|\cdot\cos|3x|\right|=\frac{1}{2};\;\; \frac{1}{2}\left|\sin|6x|\right|=\frac{1}{2}.$

$\sin|6x|=1$ Square $\sin^2|6x|=1,\;\; \cos^2|6x|=0,\;\; \cos|6x|=0,\;\; 6x=\frac{\pi}{2}+k\pi,\;\; x=\frac{\pi}{12}+\frac{k\pi}{6},\;\; k\in Z.$

Answer: $\frac{\pi}{12}+\frac{k\pi}{6},\;\; k\in Z.$

$\cos\pi x\cdot\cos|2\pi x|-\sin|\pi x|\cdot\sin 2\pi x=-1.$

Solution:

$\cos|2\pi x|=\cos 2\pi x,\;\; \cos\pi x=\cos|\pi x|.$

Taking these relations into account, this equation will have the form:

$\cos|\pi x|\cdot\cos 2\pi x-\sin|\pi x|\cdot\sin 2\pi x=-1,\;\; \cos\left(\pi|x|+2\pi x\right)=-1,$

$\pi|x|+2\pi x=\pi+2k\pi|:\pi,\;\; |x|+2x=1+2k.$

If $x\geq 0,\;\; |x|=x$ i $x+2x=1+2k,\;\; x=\frac{1+2k}{3};$

If $x<0,\;\; |x|=-x$ i $-x+2x=1+2k,\;\; x=1+2k,\;\; k\in Z.$

Answer: $\frac{1+2k}{3}, 1+2k,\;\; k\in Z.$

$tg^2 x=\frac{1-\cos|x|}{1-\sin|x|}.$

Solution:

It is known that $tg^2 x = tg^2|x|$. Then $tg^2|x| = \dfrac{1-\cos|x|}{1-\sin|x|}$. Let's introduce the substitution $|x| = y,\ \ y \ge 0.$

We get the equation $tg^2 y = \dfrac{1-\cos y}{1-\sin y}$; $\dfrac{\sin^2 y}{\cos^2 y} = \dfrac{1-\cos y}{1-\sin y}$;

Using the main property of proportion, we have:

$\sin^2 y(1-\sin y) = \cos^2 y(1-\cos y),\ \ \sin^2 y = 1-\cos^2 y,$

$(1-\cos^2 y)(1-\sin y) = (1-\sin^2 y)(1-\cos y);$

$(1-\cos^2 y)(1-\sin y) = (1-\sin y)(1+\sin y)(1-\cos y) : (1-\sin y);$

$1-\cos^2 y = (1+\sin y)(1-\cos y);\ \ \ (1-\cos^2 y) - (1+\sin y)\cdot(1-\cos y) = 0,$

$(1-\cos y)(1+\cos y) - (1+\sin y)(1-\cos y) = 0;\ \ \ (1-\cos y)\cdot(1+\cos y - 1 - \sin y) = 0,$

$$(1-\cos y)(\cos y - \sin y) = 0,\ \left[\begin{array}{l} 1-\cos y = 0, \\ \cos y - \sin y = 0 \end{array}\right. \left[\begin{array}{l} \cos y = 1, \\ tgy = 1. \end{array}\right. \left[\begin{array}{l} y = 2k\pi, \\ y = \dfrac{\pi}{4} + k\pi \end{array}\right. \ K = 0,1,2,3,...бо\ y \ge 0.$$

Then $\left[\begin{array}{l} |x| = 2k\pi, \\ |x| = \dfrac{\pi}{4} + k\pi \end{array}\right. \left[\begin{array}{l} x = \pm 2k\pi, \\ x = \pm\left(\dfrac{\pi}{4} + k\pi\right). \end{array}\right.$

Answer: $-2k\pi, 2k\pi, -\dfrac{\pi}{4} + k\pi, \dfrac{\pi}{4} + k\pi, k \in Z.$

$|\cos x| = |\cos 2x|.$

Solution:

$$\left[\begin{array}{l} \cos x = -\cos 2x, \\ \cos x = \cos 2x \end{array}\right. \left[\begin{array}{l} \cos x + \cos 2x = 0, \\ \cos x - \cos 2x = 0 \end{array}\right. \left[\begin{array}{l} 2\cos\dfrac{x+2x}{2}\cdot\cos\dfrac{2x-x}{2} = 0, \\ 2\sin\dfrac{x+3x}{2}\cdot\sin\dfrac{2x-x}{2} = 0 \end{array}\right. \left[\begin{array}{l} \cos\dfrac{3x}{2}\cdot\cos\dfrac{x}{2} = 0, \\ \sin\dfrac{3x}{2}\cdot\sin\dfrac{x}{2} = 0 \end{array}\right.$$

$$\left[\begin{array}{ll} \cos\dfrac{3x}{2} = 0, & x = \dfrac{\pi}{3} + \dfrac{2k\pi}{3},\ k \in Z \\ \cos\dfrac{x}{2} = 0, & x = \pi + 2k\pi \to x = \dfrac{2k\pi}{3} + \dfrac{\pi}{3} \\ \sin\dfrac{3x}{2} = 0, & x = \dfrac{2k\pi}{3} + 2k\pi \\ \sin\dfrac{x}{2} = 0 & x = 2k\pi \to \dfrac{2k\pi}{3}. \end{array}\right.$$

Answer: $\dfrac{2k\pi}{3};\ \dfrac{2k\pi}{3} + \dfrac{\pi}{3},\ k \in Z.$

$\sin 2x - \sin x = \sqrt{\dfrac{1-\cos x}{2}}.$

Solution:

We have: $\sin 2x - \sin x = \left|\sin\dfrac{x}{2}\right|$

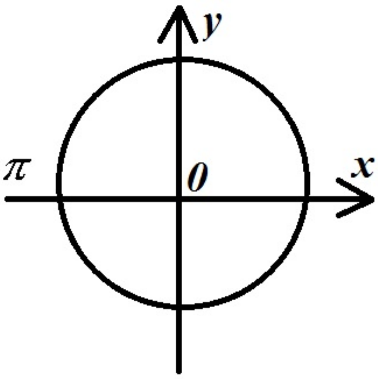
It is advisable to consider two cases:

1) $\sin\frac{x}{2} \geq 0,\ \left|\sin\frac{x}{2}\right| = \sin\frac{x}{2}.$ The equation takes the form

$\sin 2x - \sin x = \sin\frac{x}{2};\ 2\sin\frac{2x-x}{2}\cdot\cos\frac{2x+x}{2} = \sin\frac{x}{2},\ 2\sin\frac{x}{2}\cdot\cos\frac{3x}{2} - \sin\frac{x}{2} = 0;$

$\sin\frac{x}{2}\left(2\cos\frac{3x}{2} - 1\right) = 0.$

$$\left[\begin{array}{l}\sin\frac{x}{2} = 0,\\ 2\cos\frac{3x}{2} - 1 = 0\end{array}\right. \quad \left[\begin{array}{l}\sin\frac{x}{2} = 0,\\ \cos\frac{3x}{2} = \frac{1}{2}\end{array}\right. \quad \left[\begin{array}{l}\frac{x}{2} = k\pi,\\ \frac{3x}{2} = \pm\frac{\pi}{3} + 2k\pi\end{array}\right. \quad \left[\begin{array}{l}\frac{x}{2} = k\pi,\ x = 2k\pi\\ x = \frac{\pi}{9} + \frac{4\pi k}{3}\\ x = -\frac{\pi}{9} + \frac{4\pi k}{3}\end{array}\right.$$



$0 \leq \frac{x}{2} \leq \pi \quad 0 + 2\pi n \leq x \leq 2\pi + 2\pi n \quad 2k\pi,\ k \in Z.$

$\frac{\pi}{9} \in [0;\ 2\pi],\ -\frac{\pi}{9} \in (0;\ 2k\pi).$

2) $\sin\frac{x}{2} < 0,\ \left|\sin\frac{x}{2}\right| = -\sin\frac{x}{2},\ \sin 2x - \sin x = -\sin\frac{x}{2},\ 2\sin\frac{2x-x}{2}\cdot\cos\frac{2x+x}{2} = -\sin\frac{x}{2};$

$2\sin\frac{x}{2}\cdot\cos\frac{3x}{2} + \sin\frac{x}{2} = 0;\ \sin\frac{x}{2}\left(2\cos\frac{3x}{2} + 1\right) = 0;$

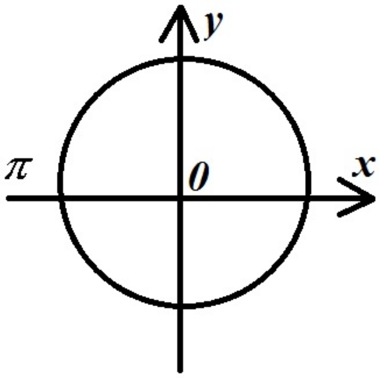
$$\left[\begin{array}{l}\sin\frac{x}{2} = 0,\\ \cos\frac{3x}{2} = -\frac{1}{2}\end{array}\right. \quad \left[\begin{array}{l}\frac{x}{2} = k\pi\\ \frac{3x}{2} = \pm\frac{2\pi}{3} + 2k\pi\end{array}\right. \quad \left[\begin{array}{l}x = 2k\pi,\\ x = \pm\frac{4}{9}\pi + \frac{4}{3}k\pi.\end{array}\right.$$

Answer: $-\frac{4\pi}{9} + 4k\pi,\ \frac{2\pi}{9} + 4k\pi,\ \frac{14\pi}{9} + 4k\pi,\ \frac{20\pi}{9} + 4k\pi,\ \frac{28\pi}{9} + 4k\pi,\ 2k\pi.$

$|\sin 5x| = \sin 5x.$

Solution:

This equality takes place when $\sin 5x \geq 0.$



$0 + 2\pi n \leq 5x \leq \pi + 2\pi n | : 5$

$\frac{2\pi n}{5} \leq x \leq \frac{\pi}{5} + \frac{2\pi n}{5},\ n \in Z.$

Answer: $\left[\frac{2\pi n}{5};\ \frac{\pi}{5} + \frac{2\pi n}{5}\right],\ n \in Z.$

$\sin^{75} x - \cos^{85} x = 1.$

Solution:

$$\left[\begin{array}{l}|\sin x| = 1,\\ |\cos x| = 1\end{array}\right. \quad \left[\begin{array}{l}\sin x = 1,\\ \cos x = -1\end{array}\right. \quad \left[\begin{array}{l}x = \dfrac{\pi}{2} + 2k\pi,\\ x = \pi + 2k\pi\end{array}\right. \quad k \in Z.$$

Answer: $\dfrac{\pi}{2} + 2k\pi;\ \pi + 2k\pi, k \in Z.$

Self-study assignments:

$2|x-3| - 5 = 0.$ Answer: 0,5; 5,5.

$|3x+1| + 5 = 0.$ Answer: $\varnothing$.

$|2|3x-1|-1| - 5 = 0.$ Answer: $-\dfrac{2}{3};\ 1\dfrac{1}{3}.$

$2|x-3| + 3x = 7.$ Answer: $1.$

$3|x+1| - |2x-1| = 5.$ Answer: $-9;\ 1.$

$|x-3| + |x-4| = 1.$ Answer: $[3; 4].$

$|x-2| + |x+3| + |x+1| - 17 = 0.$ Answer: $-6\dfrac{1}{3};\ 5.$

$\dfrac{7x+4}{5} - x = \dfrac{|3x-5|}{2}.$ Answer: 3.

$|2x-1| - |1-x| - 3 \cdot |5-3x| = -1.$ Answer: 1,4; 2.

$|x-6| + |x-12| = 6.$ Answer: $[6;\ 12].$

$|x-1| + |x-2| - |x+3| - |x-5| + 7 = 0.$ Answer: $[1;\ 2].$

$|x+1| - |x| + 3|x-1| - 2|x-2| = x + 2.$ Answer: $-2;\ [2; +\infty).$

$3x - |x+1| - |x+3| - |2x+1| - 25 = 0.$ Answer: $\varnothing$.

$3|x-2| - |3x-1| - 3|x+2| + |3x+1| + 4{,}5x = 0.$ Answer: $0; -\dfrac{4}{3}; \dfrac{4}{3}; -\dfrac{20}{9}; \dfrac{20}{9}.$

$||x-3| - |2-x|| = 1.$ Answer: $[-\infty; 2] \cup [3; +\infty).$

$2x^2 + |x| - 3 = 0.$ Answer: $-1;\ 1.$

$4x^2 + 4|x| + 1 = 0.$ Answer: $\varnothing$.

$(2x^2 + 6x)^2 - |x^2 - 2x - 3| = 17.$ Answer: $-4;\ 1.$

$|x^2 - 2x - 3| + x^2 + |2x-1| - 4 = 0.$ Answer: $\left[-1;\ \dfrac{1}{2}\right].$

$|x^2 - 5x + 6| + |x^2 - 5x + 4| = 0.$ Answer: $\varnothing$.

$|x|^3 + 3x^2 - |x| - 3 = 0.$ Answer: $-1;\ 1.$

$x^4 - 3|x|^3 + x^2 + 3|x| - 2 = 0.$ Answer: $-2; -1;\ 1;\ 2.$

$|x|^5+2x^4+2|x|^3+2x^2+2|x|=0.$ Answer: 0.

$3|x|^3+2x^2+2|x|-1=0.$ Answer: $-\frac{1}{3};\ \frac{1}{3}.$

$x^3-13x^2+36|x|=0.$ Answer: $0;4;9;\frac{13-\sqrt{313}}{2}.$

$\sqrt{x^2-4x+4}+\sqrt{x^2+4x+4}=\sqrt{x^2-6x+9}.$ Answer: $-3;-1.$

$\sqrt{x+\sqrt{6x-9}}+\sqrt{x-\sqrt{6x-9}}=\sqrt{6}.$ Answer: $[1{,}5;\ 3].$

$\sqrt{4x+8}+\sqrt{3|x|-2}=2.$ Answer: $-\frac{34}{49};2;34.$

$\sqrt{|x-2|}+|x|=10.$ Answer: $-7;\ \frac{21-\sqrt{33}}{2}.$

$\sqrt{|x+4|}+\sqrt{|x+1|}=3.$ Answer: $-5;\ 0.$

$\sqrt{x^3-2x}-|x^2-2|\cdot|x|+2=0.$ Answer: 2.

$\sqrt[3]{8|x|+4}-\sqrt[3]{8|x|-4}=2.$ Answer: $\frac{1}{2}.$

$3^{|x-4|}=\left(\frac{1}{3}\right)^{\sqrt{x}}.$ Answer: $\varnothing.$

$0{,}7^{|x-2|}=0{,}7^{\sqrt{-x^2+6x-8}}.$ Answer: 2; 3.

$3^{|2x|+1}-3^{x+1}-26=0.$ Answer: $-1;\ \log_3\left(3+\sqrt{321}\right)-\log_3 6.$

$3^{|x-1|}+2^{|x-3|}+2^{|x-3|}=448.$ Answer: $-5;9.$

$2^x=|6-x|.$ Answer: 2.

$2^{|x|}=\frac{|x|+3}{|x|+1}.$ Answer: $-1;1.$

$|\lg x|=2.$ Answer: $0{,}01;100.$

$\log_4\log_3\log_2|x|=0.$ Answer: $-8;8.$

$\lg|x+6|-\frac{1}{2}\lg|2x-3|=2-\lg 25.$ Answer: $6;14;-22\pm\sqrt{31}.$

$\sqrt{2\lg(-x)}=\lg\sqrt{x^2}.$ Answer: $-1;-100.$

$\lg\left|x-\frac{1}{2}\right|=|\lg x|-\left|\lg\frac{1}{2}\right|.$ Answer: 1.

$\sqrt{\lg(5x)\cdot\lg\left(20x^3\right)+\lg^2 2x}=3.$ Answer: $0{,}01;\ 10.$

$\log_{\frac{1}{2}}|x|=\frac{1}{4}\cdot\left(|x-2|+|x+2|\right).$ Answer: $-\frac{1}{2};\frac{1}{2}.$

$\left|\log_2|x|\right|=\frac{1}{4}\cdot\left(|x-2|+|x+2|\right).$ Answer: $-2;2;-\frac{1}{2};\frac{1}{2};-4;4.$

$\frac{\lg|4-5x|}{\lg|x|}=2.$ Answer: $4;\frac{-5-\sqrt{41}}{2};\frac{-5+\sqrt{41}}{2}.$

$\left|\sqrt{\log_{|x|}\sqrt{3x}}\cdot\log_3|x|\right|=1.$ Answer: $3;\frac{1}{9}.$

$\sqrt{\left(\sqrt{x-1}-1\right)^2}+\sqrt{\left(\sqrt{x-1}+1\right)^2}=\log_{\frac{1}{2}}(x-1).$ Answer: $\frac{5}{4}.$

Trigonometric equations with modules

$\left|\sin\left(2x-\frac{\pi}{3}\right)\right|=\frac{1}{2}.$

Solution:

This equation can be solved in two ways:
a) replace it with a set of two equations;
б) square both sides of the equation.

$\left(\left|\sin\left(2x-\frac{\pi}{3}\right)\right|\right)^2=\left(\frac{1}{2}\right)^2$, $\sin^2\left(2x-\frac{\pi}{3}\right)=\frac{1}{4}$ lower the degree of the equation:

$\frac{1-\cos\left(4x-\frac{2\pi}{3}\right)}{2}=\frac{1}{4}\Big|\cdot 2$ $\quad 1-\cos\left(4x-\frac{2\pi}{3}\right)=\frac{1}{2};$ $\quad \cos\left(4x-\frac{2\pi}{3}\right)=1-\frac{1}{2};$

$\cos\left(4x-\frac{2\pi}{3}\right)=\frac{1}{2};$ $\quad 4x-\frac{2\pi}{3}=\pm\arccos\frac{1}{2}+2\pi n,$ $\; 4x-\frac{2\pi}{3}=\pm\frac{\pi}{3}+2\pi n,$

$4x=\frac{2\pi}{3}\pm\frac{\pi}{3}+2\pi n\Big|:4$ $\; x=\frac{\pi}{6}\pm\frac{\pi}{12}+\frac{\pi n}{2},\; n\in Z.$

Answer: $\frac{\pi}{6}\pm\frac{\pi}{12}+\frac{\pi n}{2},\; n\in Z.$

Self-study assignments:

$\sin|x|=\frac{\sqrt{3}}{2}.$ Answer: $x=\pm\left((-1)+\frac{k\pi}{3}+k\pi\right),\; k=\{0;1;2;3;...\}$

$\left|\cos^2\frac{5x}{2}-\sin^2\frac{5x}{2}\right|=\frac{\sqrt{2}}{2}.$ Answer: $\frac{2k+1}{20}\pi,\; k\in Z.$

$|tgx+ctgx|=\frac{4}{\sqrt{3}}.$ Answer: $\pm\frac{\pi}{6}+\frac{k\pi}{2}\; k\in Z.$

$\left|\cos 3x\cdot\cos 5x-\frac{1}{2}\cos 2x\right|=\frac{\sqrt{2}}{4}.$ Answer: $\frac{\pi}{32}+\frac{k\pi}{16},\; k\in Z.$

$\left|\frac{tg2x-tgx}{1+tgx\cdot tg2x}\right|=2.$ Answer: $\pm arctg\alpha+k\pi,\; k\in Z.$

$\sin^2 x-\cos^2|x|=\frac{1}{2}.$ Answer: $\pm\frac{\pi}{3}+k\pi,\; k\in Z.$

$\sin|x|\cdot\cos x=-\frac{\sqrt{3}}{4}.$ Answer: $\pm\left(\frac{k\pi}{2}-(-1)^k\cdot\frac{\pi}{6}\right),\; k\in N.$

$\sin^2 x+2|\sin x|-3=0.$ Answer: $\frac{\pi}{2}+k\pi,\; k\in Z.$

$2\sin^2 x - \sin|x| - 1 = 0.$ Answer: $\pm\left((-1)^n \cdot \frac{\pi}{6} + \pi n\right), \ n \in Z.$

$\cos 2x = |\cos x|.$ Answer: $\pi n, \ n \in Z.$

$\frac{2\cos x}{|\sin x|} + tgx = 1.$ Answer: $arctg\alpha + \pi + 2\pi, \ -\frac{\pi}{4} + 2\pi n, \ n \in Z.$

$|\sin x| - \sin 3x = 0.$ Answer: $\frac{\pi}{4} + 2\pi n, \frac{3\pi}{4} + 2\pi n, -\frac{\pi}{2} + 2\pi n, \pi n, \ n \in Z.$

$|ctg 2x| = -ctg 2x.$ Answer: $\left[\frac{k\pi}{2} - \frac{\pi}{4}; \ \frac{k\pi}{2}\right), \ k \in Z.$

$|x-3|^{3x^2-10x+3} = 1.$ Answer: $\frac{1}{3}; 2; 4.$