Section 16
Trigonometric inequalities

Almost a quarter century as a trigonometric inequality ceased to be offered to entrants on the entrance exams in mathematics. Why is it so? The answer remains a mystery. However, in the last five years, some universities in Ukraine have abandoned this "fashion".

Indeed, when solving equations and their systems containing, in addition to trigonometric functions, roots or logarithms, the range of permissible values is often given by trigonometric inequalities. It is difficult to overestimate the role of trigonometric inequalities in solving exercises using the derivative of monotonicity of trigonometric functions in undefined numerical intervals.

It is advisable to recall the definition of the unit circle, sine, cosine, tangent and cotangent axes; values of trigonometric functions of some arguments.

A circle whose center is at the origin of the coordinate system and whose radius is equal to one is called a unit circle.

The vertical diameter of a unit circle is called the sine axis, the horizontal diameter is called the cosine axis. The tangent to the unit circle at a point (1; 0) called the axis of tangents, and at the point (0; 1) - axis of cotangents.

The solution of any trigonometric inequality by transformations is reduced to solving the simplest inequalities:

1). $\sin x \le a.$

а). at $|a| \le 1$ $-\arcsin a + (2\pi - 1)\cdot \pi \le x \le \arcsin a + 2\pi n, n \in Z.$

б). at $a \ge 1$ $x -$ any real number.

в). at $a < -1$ $x \in \varnothing.$

2). $\sin x \ge a.$

а). at $|a| \le 1$ $\arcsin a + 2\pi n \le x \le -\arcsin a + (2n+1)\cdot \pi, n \in Z.$

б). at $a > 1$ $x \in \varnothing.$

в). at $a < -1$ $x -$ any real number.

3). $\cos x \le a.$

а). at $|a| \le 1$ $\arccos a + 2\pi n \le x \le -\arccos a + (n+1)\cdot 2\pi, n \in Z.$

б). at $a \ge 1$ $x -$ any real number.

в). at $a < -1$ $x \in \varnothing.$

4). $\cos x \ge a.$

а). at $|a| \le 1$ $-\arccos a + 2\pi n \le x \le \arccos a + 2\pi n, n \in Z.$

б). at $a > 1$ $x \in \varnothing.$

в). at $a \le -1$ $x -$ any real number.

5). $tgx \le a.$

$$-\frac{\pi}{2} + \pi n \le x \le arctga + \pi n, n \in Z.$$

6). $tgx \ge a.$

$$arctga+\pi n\le x\le \frac{\pi}{2}+\pi n, n\in Z.$$

7). $ctgx\le a.$

$$arctga+\pi n\le x<(n+1)\pi, n\in Z.$$

8). $ctgx\ge a.$

$$\pi n< x\le arctga+\pi n, n\in Z.$$

In the process of solving trigonometric inequalities, in addition to identical transformations, it is useful to introduce an auxiliary variable.

After the inequality has been reduced to a simple one by means of identical transformations, one can complete its solution in one of three ways:

1). the use of formulas for the solution of simple trigonometric inequalities;

2). use of the unit circle;

3). using the graph of a trigonometric function, which was formed as a result of identical transformations.

Solve inequality:

$$\frac{5}{4}\sin^2 x+\frac{1}{4}\sin^2 2x>\cos 2x.$$

Solution:

Let us reduce all the terms of the inequality to one function. We apply the degree-lowering formulas on the left-hand side of the inequality:

$$\frac{5}{4}\cdot\frac{1-\cos 2x}{2}+\frac{1}{4}\cdot\left(1-\cos^2 2x\right)>\cos 2x \,\big|\times 8;$$

$$5(1-\cos 2x)+2\left(1-\cos^2 2x\right)>8\cos 2x;$$

$$5-5\cos 2x+2-2\cos^2 2x-8\cos 2x>0;$$

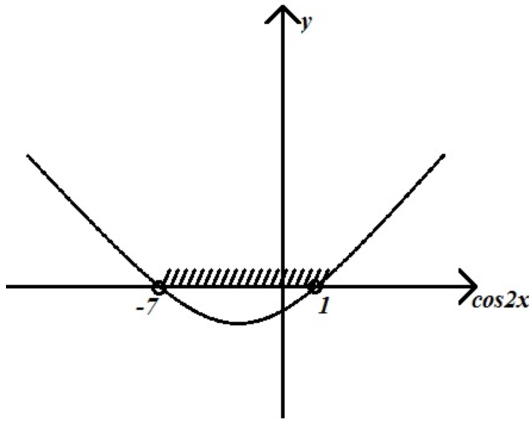
$$-2\cos^2 2x-13\cos 2x+7>0\,\big|\times(-1);$$

$$2\cos^2 2x+13\cos 2x-7<0.$$

Let's solve the square inequality with respect to $\cos 2x$:

$$D=13^2-4\cdot 2\cdot(-7)=169+56=225=15^2;$$

$$\cos 2x=\frac{-13-15}{4}=-7;\ \ \cos 2x=\frac{-13+15}{4}=\frac{1}{2}.$$



$$-7<\cos 2x<\frac{1}{2}.$$

Inequality $-7<\cos 2x$ holds for any real values of x.

We solve the inequality $\cos 2x < \frac{1}{2}$. Because $\left|\frac{1}{2}\right| \le 1$, then using the formula (3), we obtain solutions of this inequality:

$$\arccos\frac{1}{2} + 2\pi n \le 2x \le -\arccos\frac{1}{2} + 2\pi n;$$

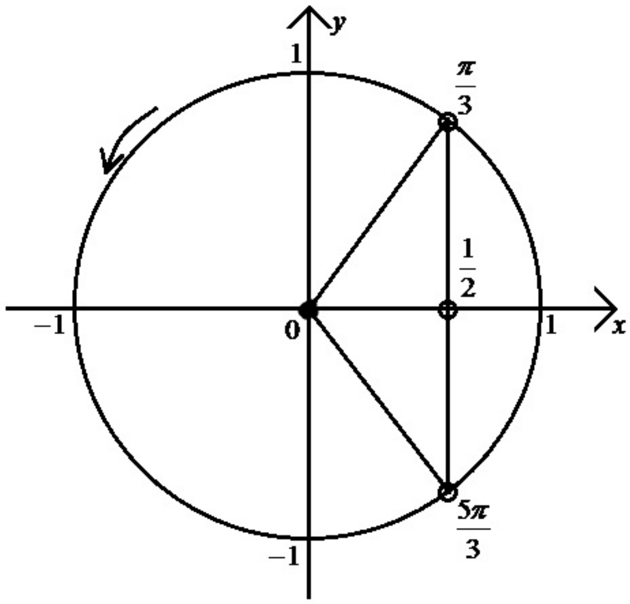
$$\frac{\pi}{3} + 2\pi n \le 2x \le -\frac{\pi}{3} + 2\pi n;$$

$$\frac{\pi}{3} + 2\pi n \le 2x \le \frac{5\pi}{3} + 2\pi n \Big| : 2;$$

$$\frac{\pi}{6} + \pi n \le x \le \frac{5\pi}{6} + \pi n, \ n \in Z.$$

Answer: $\left[\frac{\pi}{6} + \pi n; \ \frac{5\pi}{6} + \pi n\right], \ n \in Z.$

Inequality $\cos 2x < \frac{1}{2}$ can be solved with the unit circle.



A short algorithm for solving inequalities using the unit circle:
1). construct a unit circle;
2). on the cosine axis denote the number $\frac{1}{2}$ (a miniature circle in the case of a strict inequality and a shaded circle in the case of a nonstrict inequality);
3). through point $\left(\frac{1}{2}; 0\right)$ draw a chord parallel to the ordinate axis (mark the points of its intersection with the circle as indicated by the intersection point of the chord on the abscissa axis);
4). the intersection points of the chord with the circle are aligned with the origin;
5). it is fatter to point the part of the cosine axis and the part of the circle that corresponds;
6). taking into account the period of the trigonometric function, we write down the result, and after simplifying it (if necessary) - its answer (preferably in the form of a numerical interval).

The third way to solve inequality $\cos x < \frac{1}{2}$ is that:

1). build graphs of two functions $y = \cos 2x$ and $y = \frac{1}{2}$;

2). denote the points of their intersection;

3). project these points onto the abscissa axis;

4). As an intermediate result, write down the numerical interval on which the graph of the function $y = \cos 2x$ posted below the function graph $y = \frac{1}{2}$.

5). with the period of the function $y = \cos 2x$ and after identical transformations write down the answer.

$5 + 2\cos 2x \le 3 \cdot |2\sin x - 1|.$

Solution:

Reduced to one function and one argument:

$5 + 2(\cos^2 x - \sin^2 x) \le 3 \cdot |2\sin x - 1|;$

$5 + 2(1 - \sin^2 x - \sin^2 x) \le 3 \cdot |2\sin x - 1|;$

$5 + 2(1 - 2\sin^2 x) \le 3 \cdot |2\sin x - 1|;$

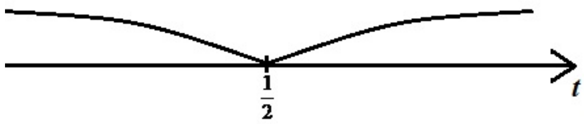
$5 + 2 - 4\sin^2 x \le 3 \cdot |2\sin x - 1|;$

$7 - 4\sin^2 x \le 3 \cdot |2\sin x - 1|;$

Let the $\sin x = t$, then:

$7 - 4t^2 \le 3 \cdot |2t - 1|;$

$2t - 1 = 0,\ 2t = 1,\ t = \frac{1}{2}.$



On $\left(-\infty; \frac{1}{2}\right)$. $2t - 1 < 0$, $|2t - 1| = 1 - 2t$, and the inequality takes the form:

$7 - 4t^2 \le 3 \cdot (1 - 2t);$

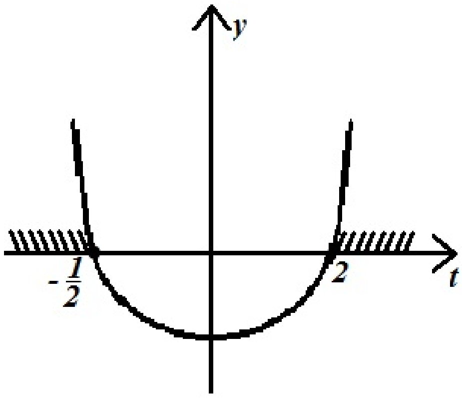
$7 - 4t^2 - 3 \cdot (1 - 2t) \le 0;$

$7 - 4t^2 - 3 + 6t \le 0;$

$-4t^2 + 6t + 4 \le 0 \,|: (-2);$

$2t^2 - 3t - 2 \ge 0;\ D = 9 + 16 = 25 = 5^2;$

$t_1 = \frac{3 - 5}{4} = -\frac{2}{4} = -\frac{1}{2};\ t_2 = \frac{3 + 5}{4} = 2.$



$\left(-\infty;-\frac{1}{2}\right]\subset\left(-\infty;\frac{1}{2}\right).$

$\left(-\infty;-\frac{1}{2}\right]-$ Solving inequality.

$(2;+\infty)\not\subset\left(-\infty;\frac{1}{2}\right).$

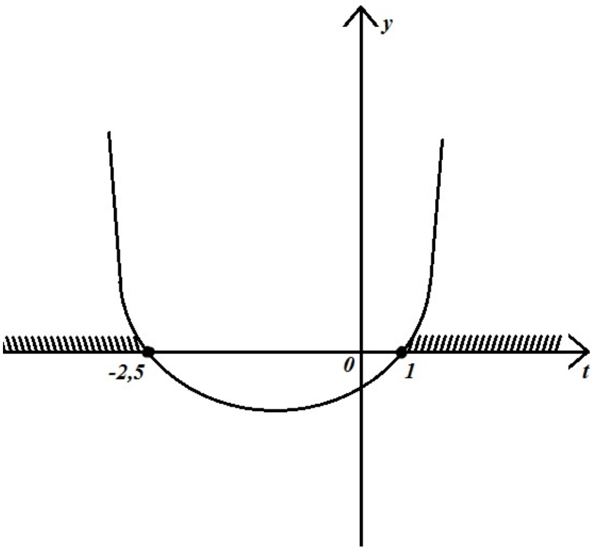
$(2;+\infty)-$ is not a solution to inequality.

On $\left[\frac{1}{2};+\infty\right)$ $2t-1>0,$ $|2t-1|=2t-1.$

$7-4t^2\le 3\cdot(2t-1);$ $7-4t^2-3\cdot(2t-1)\le 0;$

$7-4t^2-6t+3\le 0;$ $-4t^2-6t+10\le 0\,|:(-2);$

$2t^2+3t-5\ge 0.$ $D=9+40=49=7^2,$ $t_1=\frac{-3+7}{4}=1;$ $t_2=\frac{-3-7}{4}=-2{,}5.$



$y=2t^2+3t-5.$

$(-\infty;-2{,}5]\not\subset\left[\frac{1}{2};+\infty\right).$

$(-\infty;-2{,}5]$ is not a solution to inequality.

$[1;+\infty)\subset\left[\frac{1}{2};+\infty\right),$ well then $[1;+\infty)-$ is a solution to the inequality.

$\left(-\infty;-\frac{1}{2}\right]\cup[1;+\infty)-$ Solving inequality $7-4t^2\le 3\cdot|2t-1|.$

Returning to the substitution, we get:

$\sin x \le -\frac{1}{2}$. According to the formula 1) we have:

$$\arcsin\left(-\frac{1}{2}\right) + (2n-1)\pi \le x \le \arcsin\left(-\frac{1}{2}\right) + 2\pi n,$$

$$-\left(-\frac{\pi}{6}\right) + 2\pi n - \pi \le x \le -\frac{\pi}{6} + 2\pi n,$$

$$-\frac{5\pi}{6} + 2\pi n \le x \le -\frac{\pi}{6} + 2\pi n, n \in Z.$$

$\sin x \ge 1$. This inequality is correct only for x that satisfy the equation $\sin x = 1$, that is, when $x = \frac{\pi}{2} + 2\pi n, n \in Z$.

Answer: $\left[-\frac{5\pi}{6} + 2\pi n; -\frac{\pi}{6} + 2\pi n\right] \cup \left\{\frac{\pi}{2} + 2\pi n\right\}, n \in Z.$

$\sin^6 x + \cos^6 x > \frac{13}{16}.$

Solution:

The left side of this inequality requires large transformations in order for the latter to become simple.

$$\sin^6 x + \cos^6 x = (\sin^2 x)^3 + (\cos^2 x)^3 = \overbrace{\left(\underbrace{\sin^2 x + \cos^2 x}_{1}\right) \cdot (\sin^4 x - \sin^2 x \cdot \cos^2 x + \cos^4 x)}^{\text{сума кубів}} =$$

$$= \overbrace{\sin^4 x - \sin^2 x \cdot \cos^2 x + 3\sin^2 x \cdot \cos^2 x + \cos^4 x}^{\text{квадрат суми}} - 3\sin^2 x \cdot \cos^2 x =$$

$$= \left(\underbrace{\sin^2 x + \cos^2 x}_{1}\right)^2 - 3\sin^2 x \cdot \cos^2 x = 1 - 3\sin^2 x \cdot \cos^2 x =$$

$$= 1 - 3 \cdot \frac{2\sin x \cdot \cos x \cdot 2\sin x \cdot \cos x}{2 \cdot 2} = 1 - \frac{3}{4}\sin^2 2x = 1 - \frac{3}{4} \cdot \frac{1 - \cos 4x}{2} =$$

$$= 1 - \frac{3}{8} \cdot (1 - \cos 4x) = 1 - \frac{3}{8} + \frac{3}{8}\cos 4x = \frac{5}{8} + \frac{3}{8}\cos 4x.$$

Then the original inequality will have the form:

$\frac{5}{8} + \frac{3}{8}\cos 4x > \frac{13}{16}$, $\frac{3}{8}\cos 4x > \frac{13}{16} - \frac{5}{8}$, $\frac{3}{8}\cos 4x > \frac{13}{16}\Big| \times \frac{8}{3}$, $\cos 4x > \frac{1}{2}$. According to the formula (4) we have:

$$-\arccos\frac{1}{2} + 2\pi n < 4x \arccos\frac{1}{2} + 2\pi n, \quad -\frac{\pi}{3} + 2\pi n < 4x < \frac{\pi}{3} + 2\pi n, n \in Z,$$

$$-\frac{\pi}{12} + \frac{\pi}{2}n < x < \frac{\pi}{12} + \frac{\pi n}{2}, n \in Z.$$

Answer: $-\frac{\pi}{12} + \frac{\pi n}{2} < x < \frac{\pi}{12} + \frac{\pi n}{2}$, $n \in Z$; $\left(-\frac{\pi}{12} + \frac{\pi n}{2}; \frac{\pi}{12} + \frac{\pi n}{2}\right)$, $n \in Z$;

$\sin 2x + tgx \ge 2.$

Solution:

Because $tgx=\frac{\sin x}{\cos x}$, then the range of valid values $\cos x\neq 0$, that is $x\neq\frac{\pi}{2}+\pi n,\ n\in Z.$

Using the half-argument tangent formula $\sin 2x=\frac{2tgx}{1+tg^2x}$, we obtain an inequality equivalent to the original:

$\frac{2tgx}{1+tg^2x}+tgx\geq 2;$ After replacement $tgx=t$ we have:

$\frac{2t}{1+t^2}+t-2\geq 0,\left|\times\left(1+t^2\right)\right.$, бо $1+t^2>0,\ 2t+t\left(1+t^2\right)-2\left(1+t^2\right)\geq 0,\ 2t+t+t^3-2-2t^2\geq 0,$

$t^3-2t^2+3t-2\geq 0.$

$\pm 1;\ \pm 2-$ divisor of a free member.

$1:\ 1^3-2\cdot 1^2+3\cdot 1-2=0,$ and therefore 1 - root of the equation $t^3-2t^2+3t-2=0,$

$$\begin{array}{r|l} t^3-2t^2+3t-2 & t-1 \\ \underline{t^3-t^2} & t^2-t+2 \\ -t^2+3t & \\ \underline{-t^2+t} & \\ 2t-2 & \\ \underline{2t-2} & \\ 0 & \end{array}$$

$t^2-t+2=0,\ D=1-8=-7<0.$ This equation has no roots.

Inequality $t^3-2t^2+3t-2\geq 0$ equivalent to the following inequality

$(t-1)\cdot\left(\underbrace{t^2-t+2}_{>0}\right)\geq 0,$ well then $t-1\geq 0,\ t\geq 1.\ tgx\geq 1.$

$\frac{\pi}{4}+\pi n\leq x\leq\frac{\pi}{2}+\pi n,\ n\in Z.$

Answer: $\left[\frac{\pi}{4}+\pi n;\ \frac{\pi}{2}+\pi n\right),\ n\in Z.$

$\cos^2 x<\frac{3}{4}.$

Solution:

Let us apply the degree reduction formula $\cos^2 x=\frac{1+\cos 2x}{2}.$

The inequality takes the form:

$\frac{1+\cos 2x}{2}<\frac{3}{4}\left|\times 2;\right.\ 1+\cos 2x<\frac{3}{4},\ \cos 2x<\frac{1}{2}.$ According to the formula (3):

$\arccos\frac{1}{2}+2\pi n<2x<-\arccos\frac{1}{2}+2\pi(n+1);$

$\frac{\pi}{3}+2\pi n<2x<-\frac{\pi}{3}+2\pi n+2\pi;$

$\frac{\pi}{3}+2\pi n<2x<\frac{5}{3}\pi+2\pi n\left|:2;\right.$

$\frac{\pi}{6}+\pi n<x<\frac{5\pi}{6}+\pi n,\ n\in Z.$

Answer: $\left(\frac{\pi}{6}+\pi n;\ \frac{5\pi}{6}+\pi n\right), n\in Z.$

$\sin\left(\frac{\pi}{3}-2x\right)>\frac{\sqrt{3}}{2}.$

Solution:

Since sinus - an odd function, then:

$\sin\left(\frac{\pi}{3}-2x\right)=\sin\left(-\left(2x-\frac{\pi}{3}\right)\right)=-\sin\left(2x-\frac{\pi}{3}\right).$

Inequality is as follows:

$-\sin\left(2x-\frac{\pi}{3}\right)>\frac{\sqrt{3}}{2}\Big|\times(-1),\ \ \sin\left(2x-\frac{\pi}{3}\right)<-\frac{\sqrt{3}}{2},$

$-\arcsin\left(-\frac{\sqrt{3}}{2}\right)+\pi(2n-1)<2x-\frac{\pi}{3}<\arcsin\left(-\frac{\sqrt{3}}{2}\right)+2\pi n,$

$\frac{4\pi}{3}+2\pi n<2x-\frac{\pi}{3}<\frac{5\pi}{3}+2\pi n\Big|+\frac{\pi}{3};$

$\frac{5\pi}{3}+2\pi n<2x<2\pi+2\pi n\Big|:2;$

$\frac{5\pi}{6}+\pi n<x<\pi+\pi n, n\in Z.$

Answer: $\left(\frac{5\pi}{6}+\pi n;\pi+\pi n\right), n\in Z.$

$\cos^2\left(\frac{3x}{2}+\frac{\pi}{12}\right)>\frac{3}{4}.$

Solution:

$\frac{1+\cos\left(3x+\frac{\pi}{6}\right)}{2}>\frac{3}{4};\ 1+\cos\left(3x+\frac{\pi}{6}\right)>\frac{3}{2};\ \cos\left(3x+\frac{\pi}{6}\right)>\frac{1}{2};$

$-\arccos\frac{1}{2}+2\pi n<3x+\frac{\pi}{6}<\arccos\frac{1}{2}+2\pi n;$

$-\frac{\pi}{3}+2\pi n<3x+\frac{\pi}{6}<\frac{\pi}{3}+2\pi n\Big|-\frac{\pi}{6};$

$-\frac{\pi}{2}+2\pi n<3x<\frac{\pi}{6}+2\pi n\Big|:3;$

$-\frac{\pi}{6}+\frac{2\pi n}{3}<x<\frac{\pi}{18}+\frac{2\pi n}{3}.$

Answer: $\left(-\frac{\pi}{6}+\frac{2\pi n}{3};\frac{\pi}{18}+\frac{2\pi n}{3}\right), n\in Z.$

$\sin\frac{\pi}{x-1}<\frac{1}{\sqrt{2}}.$

Solution:

$\sin\frac{\pi}{x-1}<\frac{\sqrt{2}}{2};\ -\arcsin\frac{\sqrt{2}}{2}+\pi(2n-1)<\frac{\pi}{x-1}<\arcsin\frac{\sqrt{2}}{2}+2\pi n,$

$\frac{3\pi}{4}+2k\pi<\frac{\pi}{x-1}<\frac{9\pi}{4}+2k\pi\,\Big|:\pi;$

$\frac{3}{4}+2k<\frac{1}{x-1}<\frac{9}{4}+2k;\ \frac{3+8k}{4}<\frac{1}{x-1}<\frac{9+8k}{4};$

"We flip" double inequality: 1). $k=-1;$

$\frac{4}{8k+9}<x-1<\frac{4}{3+8k}\Big|+1;\ 1+\frac{4}{8k+9}<x<\frac{4}{3+8k}+1;\ \frac{8k+9+4}{8k+9}<x<\frac{4+8k+3}{3+8k};$

$\frac{8k+13}{8k+9}<x<\frac{8k+7}{3+8k};$

2). At $k=-1;$

$-\frac{5}{4}<\frac{1}{x-1}<\frac{1}{4},\ \frac{1}{x-1}\neq 0;\ -\frac{5}{4}<\frac{1}{x-1}<0;\ 8<x-1<-\frac{4}{5};\ 4<x-1<\infty,\ x<\frac{1}{5}$ or $x>5.$

Answer: $x<\frac{1}{5},\ x>5,$ at $k-1.$

$\frac{8k+13}{8k+9}<x<\frac{8k+7}{8k+3},$ at $k\neq -1.$

$\cos\left(2\pi\cdot\sin\frac{\pi x}{2}\right)<-\frac{1}{2}.$

Solution:

$\frac{2\pi}{3}+2k\pi<2\pi\cdot\sin\frac{\pi x}{2}<\frac{4\pi}{3}+2n\pi,\Big|:2\pi$

$\frac{1}{3}+n<\sin\frac{\pi x}{2}<\frac{2}{3}+n.$ Because $\left|\sin\frac{\pi x}{2}\right|\leq 1,$ then for this inequality to be valid it is necessary that $n=0,\ \frac{1}{3}<\sin\frac{\pi x}{2}<\frac{2}{3}$ (A) or

$n=-1,\ \frac{1}{3}-1<\sin\frac{\pi x}{2}<\frac{2}{3}-1,\ -\frac{2}{3}<\sin\frac{\pi x}{2}<-\frac{1}{3}$ *(B)*

We raise the inequality (A) in a square:

$\frac{1}{9}<\sin^2\frac{\pi x}{2}<\frac{4}{9};\ \frac{1}{9}<\frac{1-\cos\pi x}{2}<\frac{4}{9}\Big|\times 2;\ \frac{2}{9}<1-\cos\pi x<\frac{8}{9}\Big|(-1);\ \frac{2}{9}-1<-\cos\pi x<\frac{8}{9}-1;$

$-\frac{7}{9}<-\cos\pi x<-\frac{1}{9}\Big|\times(-1);\ \frac{7}{9}>\cos\pi x>\frac{1}{9};\ \frac{1}{9}<\cos\pi x<\frac{7}{9};$ *(C)*

Since the I and IV quarters have the function $y=\cos x$ decreases, then the solution to inequality C is $\arccos\frac{7}{9}+2\pi n<\pi x<\arccos\frac{1}{9}+2\pi n\,\Big|:\pi,$

$\frac{1}{\pi}\arccos\frac{7}{9}+2n<x<\frac{1}{\pi}\arccos\frac{1}{9}+2n,\ x\in\left(\frac{1}{\pi}\arccos\frac{7}{9}+2n;\ \frac{1}{\pi}\arccos\frac{1}{9}+2\pi n\right), n\in Z.$

Let us solve the equation B, lift it into the square.

$\frac{1}{9}<\sin^2\frac{\pi x}{2}<\frac{4}{9};\ \frac{1}{9}<1-\cos\pi x<\frac{4}{9};\ \arccos\left(-\frac{1}{9}\right)+2\pi n<\pi x<\arccos\left(-\frac{7}{9}\right)+2\pi n,$

$2n+\frac{1}{\pi}\arccos\left(-\frac{1}{9}\right)<x<\frac{1}{\pi}\arccos\left(-\frac{7}{9}\right)+2n.$

Answer: $\left(\frac{1}{\pi}\arccos\frac{7}{9}+2n;\frac{1}{\pi}\arccos\frac{1}{9}+2n\right);\ \left(2n+\frac{1}{\pi}\arccos\left(-\frac{1}{9}\right);2n+\frac{1}{\pi}\arccos\left(-\frac{7}{9}\right)\right).$

$2\cos^2 x - \sin x > 1.$

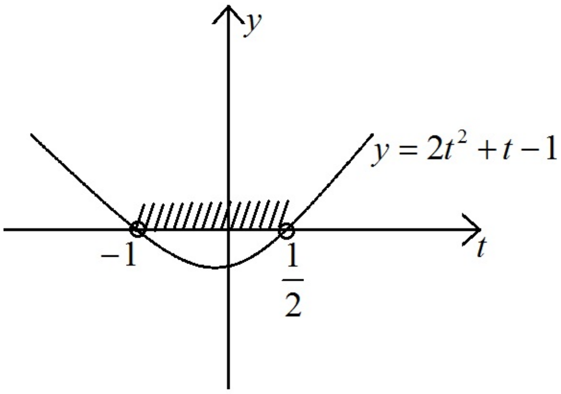
Solution:

Let the $\sin x = t,\ |t| \le 1,\ \cos^2 x = 1 - \sin^2 x = 1 - t^2.$

The original inequality becomes:

$2\cdot(1-t^2) - t > 1;\ 2 - 2t^2 - t - 1 > 0;\ -2t^2 - t + 1 > 0 \,|\times(-1);$

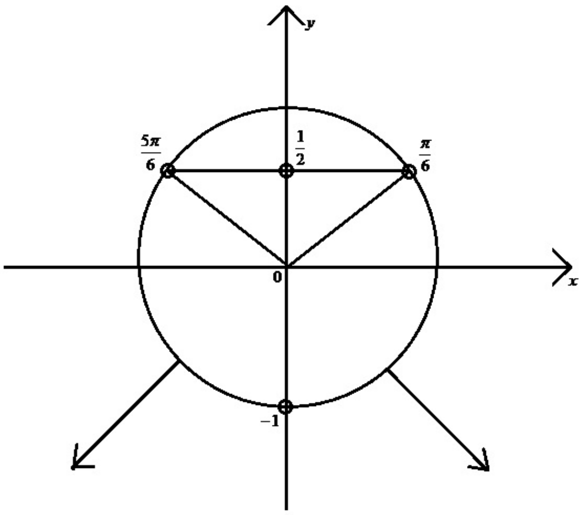
$2t^2 + t - 1 < 0.\ D = 1 + 8 = 9,\ t_1 = \dfrac{-1-3}{2\cdot 2} = -1,\ t_2 = \dfrac{-1+3}{2\cdot 2} = \dfrac{1}{2}.$



$t \in \left(-1; \dfrac{1}{2}\right)$, which does not contradict the condition $|t| \le 1.$

Returning to the substitution, we obtain the double inequality $-1 < \sin x < \dfrac{1}{2}$, equivalent to a system of inequalities:

$$\begin{cases} \sin x > -1, \\ \sin x < \dfrac{1}{2}. \end{cases}$$



$\dfrac{5\pi}{6} + 2\pi n < x < \dfrac{3\pi}{2} + 2\pi n,\ \dfrac{3\pi}{2} + 2\pi n < x < \dfrac{13\pi}{6} + 2\pi n.$

Answer: $\left(\dfrac{5\pi}{6} + 2\pi n; \dfrac{3\pi}{2} + 2\pi n\right), n \in Z;\ \left(\dfrac{3\pi}{2} + 2\pi n; \dfrac{13\pi}{6} + 2\pi n\right), n \in Z.$

$tg\dfrac{\pi x}{2} - ctg\dfrac{\pi x}{2} > \dfrac{2}{\sqrt{3}}.$

Solution:

Let's introduce a new variable $tg\frac{\pi x}{2}=t,$ then $ctg\frac{\pi x}{2}=\frac{1}{t}.$

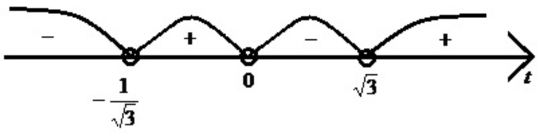
The inequality takes the form:

$$t-\frac{1}{t}>\frac{2}{\sqrt{3}}\to t-\frac{1}{t}-\frac{2}{\sqrt{3}}>0\to\frac{t^2\sqrt{3}-\sqrt{3}-2t}{t\sqrt{3}}>0;\ \frac{\sqrt{3}t^2-2t-\sqrt{3}}{t\sqrt{3}}>0,t\neq 0.$$

In the domain of definition, this inequality is equivalent to:

$$\frac{1}{\sqrt{3}}\cdot t\cdot\left(\sqrt{3}t^2-2t-\sqrt{3}\right)>0,\ \ D=4+4\cdot 3=16=4^2.$$

$$t_1=\frac{2-4}{2\sqrt{3}}=-\frac{2}{2\sqrt{3}}=-\frac{1}{\sqrt{3}};\ \ t_2=\frac{2+4}{2\sqrt{3}}=\frac{6}{2\sqrt{3}}=\frac{3}{\sqrt{3}}=\sqrt{3};\ \ t_3=0.$$



$$t\cdot\left(t+\frac{1}{\sqrt{3}}\right)\cdot\left(t-\sqrt{3}\right)>0.$$

$$-1\cdot\left(-1+\frac{1}{\sqrt{3}}\right)\left(-1-\sqrt{3}\right)="-"\cdot"-"\cdot"-"="-";\ \ 1\left(1+\frac{1}{\sqrt{3}}\right)\left(1-\sqrt{3}\right)="-";$$

$$-\frac{1}{10}\cdot\left(-\frac{1}{10}+\frac{1}{\sqrt{3}}\right)\left(-\frac{1}{10}-\sqrt{3}\right)="-"\cdot"+"\cdot"-"="+";\ \ 2\left(2+\frac{1}{\sqrt{3}}\right)\left(2-\sqrt{3}\right)="+";$$

$t\in\left(-\frac{1}{\sqrt{3}};0\right)\cup\left(\sqrt{3};+\infty\right)$ Using substitution.

$$\left[\begin{array}{l}-\frac{1}{\sqrt{3}}<tg\frac{\pi x}{2}<0,\\ tg\frac{\pi x}{2}>3.\end{array}\right.\left[\begin{array}{l}-\frac{\pi}{6}+\pi n<\frac{\pi x}{2}<\pi n,n\in Z.\\ \frac{\pi}{3}+\pi m<\frac{\pi x}{2}<\frac{\pi}{2}+\pi m,m\in Z.\end{array}\right.$$

$$\left[\begin{array}{l}-\frac{\pi}{3}+\pi n<\pi x<2\pi n\ \ |:\pi\\ \frac{2\pi}{3}+2\pi m<\pi x<\pi+2\pi m\ \ |:\pi.\end{array}\right.\qquad\left[\begin{array}{l}-\frac{1}{3}+2n<x<2n,n\in Z.\\ \frac{2}{3}+2m<x<1+2m.\end{array}\right.$$

Answer: $\left(-\frac{1}{3}+2n;\ 2n\right)\cup\left(\frac{2}{3}+2m;\ 1+2m\right),\ n\in Z,\ m\in Z.$

$\sin^2 x-3\sin x\cdot\cos x+2\cos^2 x<0.$

Solution:

Suppose that $\cos x=0,$ then $\sin^2 x-3\sin x\cdot 0+2\cdot 0^2=\sin^2 x.$

Taking into account the sign of the original inequality, we arrived at the absurd inequality $\sin^2 x<0.$ Well then, $\cos x\neq 0.$ Well then, $\cos^2 x>0.$

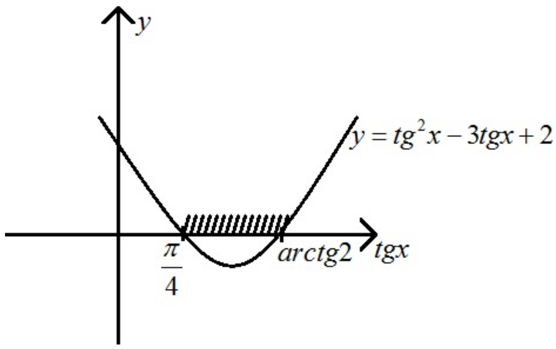
We divide both sides of the inequality by $\cos^2 x$:

$$\frac{\sin^2 x}{\cos^2 x}-\frac{3\sin x\cdot\cos x}{\cos^2 x}+\frac{2\cos^2 x}{\cos^2 x}<0;$$

$$tg^2x-3tgx+2<0.$$

The roots of the right-hand side of the inequality $tgx=1,\ tgx=2.$

$x = \frac{\pi}{4}, \;\; x = arctg2$



$\frac{\pi}{4} + \pi n < x < arctg2 + \pi n, \;\; n \in Z.$

Answer: $\left(\frac{\pi}{4} + \pi n; \; arctg2 + \pi n\right), \;\; n \in Z.$

$4\sin^2 x - 3\sin x \cdot \cos x + 3\cos^2 x > 2.$

Solution:

$2 = 2 \cdot 1 = 2 \cdot \left(\sin^2 x + \cos^2 x\right);$

$4\sin^2 x - 3\sin x \cdot \cos x + 3\cos^2 x > 2\left(\sin^2 x + \cos^2 x\right);$

$4\sin^2 x - 3\sin x \cdot \cos x + 3\cos^2 x - 2\sin^2 x - 2\cos x > 0;$

$2\sin^2 x - 3\sin x \cdot \cos x + \cos^2 x > 0;$

If $\cos x = 0,$ then $2\sin^2 x - 3\sin x \cdot 0 + 0^2 = 2\sin^2 x.$

$2\sin^2 x > 0 -$ correct inequality, and therefore divide both sides of the inequality $\cos^2 x$ not allowed.

From the condition $\cos x = 0$ it follows that $x = \frac{\pi}{2} + k\pi, \;\; k \in Z.$

Let us find the remaining solutions to the inequality. They satisfy the condition $x \neq \frac{\pi}{2} + \pi n,$ then $\cos x \neq 0,$ a $\cos^2 x > 0.$

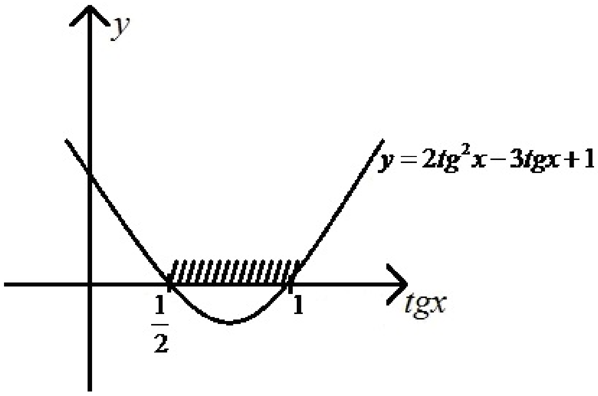
Dividing both sides of the inequality by $\cos^2 x,$ we get:

$\frac{2\sin^2 x}{\cos^2 x} - \frac{3\sin x \cdot \cos x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} > \frac{0}{\cos^2 x};$

$2tg^2 x - 3tgx + 1 > 0.$

We will solve this square inequality in a graphical way.:

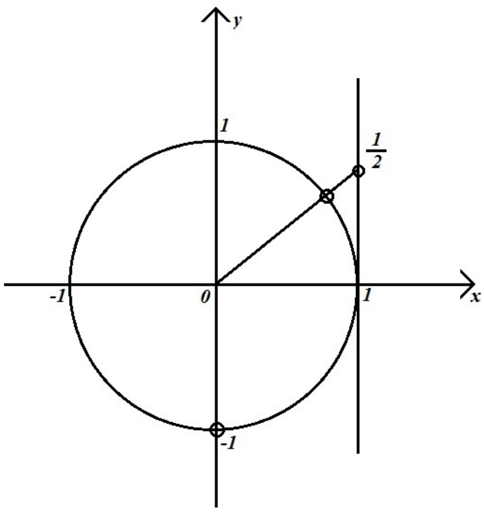
$D = 9 - 8 = 1; \;\; tgx = \frac{3-1}{4} = \frac{1}{2}; \;\; tgx = \frac{3+1}{4} = 1.$

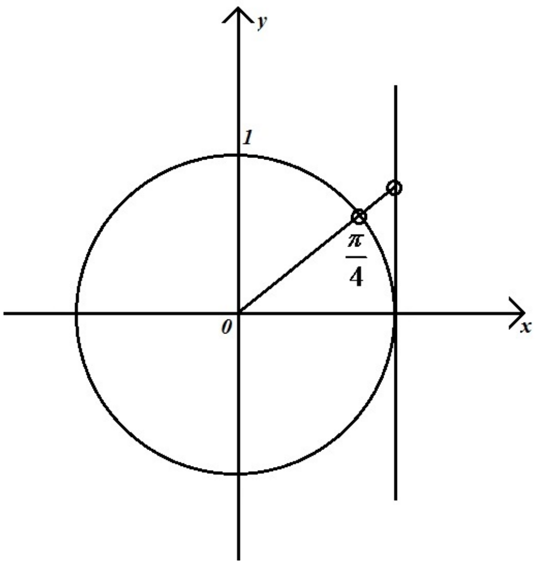


$$-\frac{\pi}{2} + \pi n < x < arctg\frac{1}{2} + \pi n, \quad n \in Z.$$

Let us solve the set of inequalities:

$$\left[\begin{array}{l} tgx < \frac{1}{2}, \\ tgx > 1. \end{array}\right.$$





$$\frac{\pi}{4} + \pi n < x < \frac{\pi}{2} + \pi n, n \in Z.$$

Answer: $\left(-\frac{\pi}{2} + \pi n;\ arctg\frac{1}{2} + \pi n\right) \cup \left(\frac{\pi}{4} + \pi n;\ \frac{\pi}{2} + \pi n\right).$

Given a group of roots $x=\frac{\pi}{2}+k\pi,$ The answer can be written like this:

$$\left(-\frac{\pi}{2}+\pi n;\ arctg\frac{1}{2}+\pi n\right)\cup\left(\frac{\pi}{4}+\pi n;\ \frac{\pi}{2}+\pi n\right],\ n\in Z.$$

$\cos x-\sin 2x-\cos 3x<0.$

Solution:

Inequalities of this kind are addressed in specific ways. Consider one of them.
Let we have a function $f(x)=\cos x-\sin 2x-\cos 3x.$
Since the period for the function $y=\cos x$ is $T=2\pi,$ for $y=\sin 2x$ is $T=\pi,$ for $y=\cos 3x$ is $T=\frac{2\pi}{3},$ then the period for the function $f(x)\in 2\pi.$
Let us solve this inequality on the interval $[0;2\pi)$:

$$f(x)=\cos x-\sin 2x-\cos 3x=(\cos x-\cos 3x)-\sin 2x=2\sin\frac{x+3x}{2}\cdot\sin\frac{3x-x}{2}-\sin 2x=$$
$$=2\sin 2x\cdot\sin x-\sin 2x=\sin 2x\cdot(2\sin x-1).$$

Let us find the roots of this function on the interval $[0;2\pi)$:

$$f(x)=0;\ \sin 2x\cdot(2\sin x-1)=0,\ \left[\begin{array}{l}\sin 2x=0,\\ 2\sin x-1=0.\end{array}\right.$$

$2x=\pi n,\ \ x=\frac{\pi n}{2}:$

$n=0, x=0,\ \sin x=\frac{1}{2};\ x=(-1)^n\cdot\frac{\pi}{6}+\pi n.$

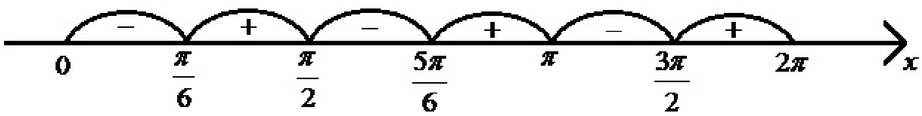
$n=1, x=\frac{\pi}{2},\ n=0,\ x=\frac{\pi}{6}.$

$n=2, x=\pi,\ n=1,\ x=-\frac{\pi}{6}+\pi=\frac{5\pi}{6}.$

$n=3, x=\frac{3\pi}{2},$

Found roots of the function $f(x)$: $0,\frac{\pi}{6},\frac{\pi}{2},\frac{5\pi}{6},\pi,\frac{3\pi}{2}.$

They share the interval $[0;2\pi)$ 6 intervals of constant sign:



On $\left[0;\frac{\pi}{6}\right)$ $\sin 2x>0,\ 2\sin x-1<0,\ f(x)<0.$

On $\left[\frac{\pi}{6};\frac{\pi}{2}\right)$ $\sin 2x>0,\ 2\sin x-1>0,\ f(x)>0.$

On $\left(\frac{\pi}{2};\frac{5\pi}{6}\right)$ $\frac{\pi}{2}<x<\frac{5\pi}{6},\ \pi<2x<\frac{5\pi}{3},\ \sin 2x<0,\ 2\sin x-1>0,\ f(x)<0.$

On $\left(\frac{5\pi}{6};\pi\right)$ $\frac{5\pi}{6}<x<\pi,\ \frac{5\pi}{3}<2x<\frac{10\pi}{3},\ \sin 2x<0,\ 2\sin x-1<0,\ f(x)>0.$

On $\left(\pi;\ \frac{3\pi}{2}\right)$ $2\pi < 2x < 3\pi,\ \sin 2x > 0,\ 2\sin x - 1 < 0,\ f(x) < 0.$

On $\left(\frac{3\pi}{2};\ 2\pi\right)$ $3\pi < 2x < 4\pi,\ \sin 2x < 0,\ 2\sin x - 1 < 0,\ f(x) > 0.$

On $[0; 2\pi)$ the original inequality has solutions:

$$x \in \left(0; \frac{\pi}{6}\right) \cup \left(\frac{\pi}{2}; \frac{5\pi}{6}\right) \cup \left(\pi; \frac{3\pi}{2}\right).$$

Given the sine period $T = 2\pi,$ we write down all solutions to the inequality:

$$x \in \left(2\pi n; \frac{\pi}{6} + 2\pi n\right) \cup \left(\frac{\pi}{2} + 2\pi n; \frac{5\pi}{6} + 2\pi n\right) \cup \left(\pi + 2\pi n; \frac{3\pi}{2} + 2\pi n\right), n \in Z.$$

Answer: $\left(2\pi n; \frac{\pi}{6} + 2\pi n\right) \cup \left(\frac{\pi}{2} + 2\pi n; \frac{5\pi}{6} + 2\pi n\right) \cup \left(\pi + 2\pi n; \frac{3\pi}{2} + 2\pi n\right), n \in Z.$

$ctgx > ctg3x.$

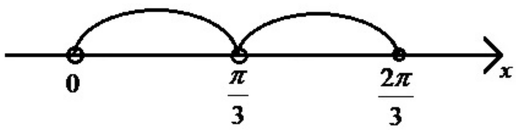
Solution:

$ctgx - ctg3x > 0.\ \ f(x) = ctgx - ctg3x,\ T = \pi.$

Let us solve this inequality for $[0; \pi)$:

On this interval, the inequality is defined for all values of x, except $0;\ \frac{\pi}{3};\ \frac{2\pi}{3}$

$\left(ctg0 \text{ не існує}, ctg\left(3\cdot\frac{\pi}{3}\right) \text{не існує}, ctg\left(3\cdot\frac{2\pi}{3}\right) \text{не існує}\right).$



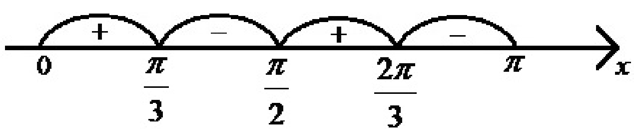
Find the zeros of the function $f(x) = ctgx - ctg3x$:

$$f(x) = \frac{\cos x}{\sin x} - \frac{\cos 3x}{\sin 3x} = \frac{\sin 3x\cdot\cos x - \cos 3x\cdot\sin x}{\sin x\cdot\sin 3x} = \frac{\sin(3x - x)}{\sin x\cdot\sin 3x} = \frac{\sin 2x}{\sin x\cdot\sin 3x};$$

$$f(x) = 0;\ \ \frac{\sin 2x}{\sin x\cdot\sin 3x} = 0;\ \ \begin{cases}\sin 2x = 0,\\ \sin x \neq 0,\\ \sin 3x \neq 0.\end{cases}$$

$2x = \pi;\ \ x = \frac{\pi}{2}$ – the only zero in $[0; \pi)$.

Thus, the function $f(x) = \frac{\sin 2x}{\sin x\cdot\sin 3x}$ on the interval $[0; \pi)$ has such intervals of constancy:



Let us define the sign of the function $f(x)$ on each of them:

$$f(x) = \frac{2\sin x\cdot\cos x}{\sin x\cdot\sin 3x} = \frac{2\cos x}{\sin 3x}.$$

On $\left[0;\dfrac{\pi}{3}\right)$ $\quad\cos x>0,\ 0<x<\dfrac{\pi}{3},\ 0<3x<\pi,\ \sin 3x>0,\ f(x)>0.$

On $\left(\dfrac{\pi}{3};\dfrac{\pi}{2}\right)$ $\quad\cos x>0,\ \dfrac{\pi}{3}<x<\dfrac{\pi}{2}\Big|\cdot 3\ \ \pi<3x<\dfrac{3\pi}{2},\ \sin 3x<0,\ f(x)<0.$

On $\left(\dfrac{\pi}{2};\dfrac{2\pi}{3}\right)$ $\quad\dfrac{\pi}{2}<x<\dfrac{2\pi}{3}\Big|\cdot 3\ \ \dfrac{3\pi}{2}<3x<2\pi,\ \cos x<0,\ \sin 3x<0,\ f(x)>0.$

On $\left(\dfrac{3\pi}{3};\pi\right)$ $\quad\cos x<0,\ \dfrac{2\pi}{3}<x<\pi\Big|\cdot 3\ \ 2\pi<3x<3\pi,\ \sin 3x>0,\ f(x)<0.$

We choose the intervals at which $f(x)>0$:

$\left[0;\dfrac{\pi}{3}\right)\cup\left(\dfrac{\pi}{2};\dfrac{2\pi}{3}\right)$. Adding a function period $f(x)$ $T=\pi$:

Answer: $\left(k\pi;\dfrac{\pi}{3}+k\pi\right)\cup\left(\dfrac{\pi}{2}+k\pi;\dfrac{2\pi}{3}+k\pi\right), k\in Z.$

$$\cos^3 x\cdot\cos 3x-\sin^3 x\cdot\sin 3x>\frac{5}{8}.$$

Solution:

By the formula of the triple argument, we have:

$$\cos 3x=4\cos^3 x-3\cos x,\ \ \sin 3x=3\sin x-4\sin^3 x,$$

$$4\cos^3 x=\cos 3x+3\cos x,\ \ 4\sin^3 x=3\sin x-\sin 3x,$$

$$\cos^3 x=\frac{\cos 3x+3\cos x}{4}.\quad \sin^3 x=\frac{3\sin x-\sin 3x}{4}.$$

Then the original inequality will have the form:

$$\frac{\cos 3x+3\cos x}{4}\cdot\cos 3x-\frac{3\sin x-\sin 3x}{4}\cdot\sin 3x>\frac{5}{8}\ \ \Big|\cdot 4$$

$$(\cos 3x+3\cos x)\cdot\cos 3x-(3\sin x-\sin 3x)\cdot\sin 3x>\frac{5}{2},$$

$$\cos^2 3x+3\cos x\cdot\cos 3x-3\sin x\cdot\sin 3x+\sin^2 3x>\frac{5}{2},$$

$$\underbrace{\cos^2 3x+\sin^2 3x}_{1}+3\cdot(\cos x\cdot\cos 3x-\sin x\cdot\sin 3x)>\frac{5}{2},$$

$$1+3\cdot\cos(x+3x)>\frac{5}{2},$$

$$3\cos 4x>2{,}5-1,$$

$$\cos 4x>\frac{1{,}5}{3},$$

$$\cos 4x>\frac{1}{2},$$

$$-\frac{\pi}{3}+2\pi n<4x<\frac{\pi}{3}+2\pi n\ \ \Big|:4$$

$$-\frac{\pi}{12}+\frac{\pi n}{2}<x<\frac{\pi}{12}+\frac{\pi n}{2},\ \ n\in Z.$$

Answer: $\left(-\dfrac{\pi}{12}+\dfrac{\pi n}{12};\dfrac{\pi}{12}+\dfrac{\pi n}{2}\right), n\in Z.$

Self-study assignments:

$\sin x > -\frac{1}{2}.$ Answer: $\left(-\frac{\pi}{6}+2\pi n; \frac{7\pi}{6}+2\pi n\right), n \in Z.$

$\cos x \le -\frac{1}{2}.$ Answer: $\left(\frac{2\pi}{3}+2\pi n; \frac{4\pi}{3}+2\pi n\right), n \in Z.$

$tgx < 2.$ Answer: $\left(-\frac{\pi}{2}+\pi n;\ arctg2+\pi n\right), n \in Z.$

$\sin\left(\frac{3}{2}x+\frac{\pi}{12}\right) > \frac{1}{\sqrt{2}}.$ Answer: $\left(\frac{4}{9}\pi+\frac{4}{3}\pi n;\ \frac{13}{9}\pi+\frac{4}{3}\pi n\right), n \in Z.$

$\cos 2x - \sin 3x \ge 0.$ Answer: $\left[-\frac{3}{8}\pi+\pi n;\ \frac{\pi}{8}+\pi n\right], n \in Z.$

$\sin x + \cos 2x > 1.$ Answer: $\left(2\pi n;\ \frac{\pi}{6}+2\pi n\right),\ n \in Z.$

$\sin 3x \cdot \cos x + \cos x \cdot \sin x \ge \frac{1}{2}.$ Answer: $\left[\frac{\pi}{24}+\frac{\pi n}{2};\ \frac{5\pi}{24}+\frac{\pi n}{2}\right],\ n \in Z.$

$6\sin^2 x - 5\sin x + 1 \ge 0.$ Answer: $\left[\frac{\pi}{6}+2\pi n;\ \frac{5\pi}{6}+2\pi n\right],\ n \in Z.$

$\frac{15}{\sin x + 1} < 11 - 2\sin x.$ Answer: $\left(\frac{\pi}{6}+2\pi n;\ \frac{5\pi}{6}+2\pi n\right),\ n \in Z.$

$\sin^2\left(x-\frac{\pi}{6}\right) < \frac{1}{4}.$ Answer: $\left(k\pi;\ \frac{\pi}{3}+k\pi\right),\ k \in Z.$

$2-\frac{5}{2}-\sin x > \cos^2 x.$ Answer: $\left(-\frac{7\pi}{6}+2\pi n;\ \frac{\pi}{6}+2\pi n\right),\ n \in Z.$

$2\sin 2x < 5\cdot(1+\cos x - \sin x).$ Answer: $\left(-\pi+2k\pi;\ \frac{\pi}{2}+2k\pi\right),\ k \in Z.$

$\frac{\sin x + \cos x - 1}{\sin x - \cos x + 1} > 0.$ Answer: $\left(-\frac{\pi}{2}+2k\pi;\ 2k\pi\right) \cup \left(2k\pi;\ \frac{\pi}{2}+2k\pi\right),\ k \in Z.$

$\sin 2x < \sin 3x.$

Answer: $\left(2k\pi;\ \frac{\pi}{5}+2k\pi\right) \cup \left(\frac{3}{5}\pi+2k\pi; \pi+2k\pi\right) \cup \left(\frac{7}{5}\pi+2k\pi; \frac{9}{5}\pi+2\pi\right).$

$\sin\frac{2-x}{3} > \frac{1}{2}.$ Answer: $(6k\pi+3{,}5\pi+2; 6k\pi+5{,}5\pi+2),\ k \in Z.$

$\cos\sqrt{1-x} < \frac{1}{\sqrt{2}}.$ Answer: $\left(1-\left(2k\pi+\frac{7\pi}{4}\right)^2; 1-\left(2k\pi+\frac{\pi}{4}\right)^2\right),\ k \in Z.$

$tg\frac{\pi x}{3(x-1)} > \sqrt{3}.$ Answer: $(3;+\infty).$

$\frac{2\sin^2 x + \sin x - 1}{\sin x - 1} > 0.$ Answer: $\left(\frac{\pi}{2}(4n-1); \frac{\pi}{6}(12n+1)\right), n \in Z.$

$2\cos 2x + \sin 2x \ge tgx.$ Answer: $\left(-\frac{\pi}{2}+\pi n; -arctg2+\pi n\right) \cup \left[-\frac{\pi}{4}+\pi n; \frac{\pi}{4}+\pi n\right], n \in Z.$

$4\sin\frac{x}{2}\cdot\cos\frac{x}{2}\le -1.$ Answer: $\left[\frac{7\pi}{6}+2\pi n;\ \frac{11\pi}{6}+2\pi n\right],\ n\in Z.$

$ctg\left(\frac{3\pi}{2}-\frac{x}{2}\right)\le\sqrt{3}.$ Answer: $\left[-\pi+\pi n;\ \frac{2\pi}{3}+\pi n\right],\ n\in Z.$