# Section 4
# Converting trigonometric expressions

The transformation of trigonometric expressions, like the transformation of any other expressions, is a reliable propaedeutics for the further solution of trigonometric equations and their systems; proof of identities, etc.

Since this material constitutes a significant part of the entire volume of mathematical material of a general education school, it is necessary to pay great attention to the transformation of trigonometric expressions.

It is very important to choose a system of exercises that would contribute to the development of skills and abilities to calculate the values of trigonometric functions of any values of the argument by the known value of one of them.

Some methodologists, to make it easier to memorize trigonometric formulas and navigate in them, recommend dividing the basic trigonometric identities into three groups:

**1). Square formulas:**

$\sin^2 y + \cos^2 y = 1,$

$\sec^2\alpha - tg^2\alpha = 1, \qquad \sec\alpha = \dfrac{1}{\cos\alpha};$

$\cos ec^2\alpha - ctg^2\alpha = 1, \qquad \cos ec\alpha = \dfrac{1}{\sin\alpha}.$

**2). Formula reciprocals:**

$\sin\alpha \cdot \cos ec\alpha = 1,$

$\cos\alpha \cdot \sec\alpha = 1,$

$tg\alpha \cdot ctg\alpha = 1.$

**3). Division Formulas:**

$\dfrac{\sin\alpha}{\cos\alpha} = tg\alpha, \qquad \dfrac{\cos\alpha}{\sin\alpha} = ctg\alpha.$

Groups of formulas should adhere to the following sequence: first use the formulas of squares, then the formulas of reciprocals, after them - the formulas for division and, finally, again the formulas of reciprocals. Let's illustrate this tip by solving this exercise.

Given: $\sec\alpha = -\dfrac{25}{7},\ 180^\circ \langle \alpha \ \langle 270^\circ$

Calculate the value of the other five trigonometric functions.

Solution:

From formulas squares choose $\sec^2\alpha - tg^2\alpha = 1.$ We define $tg^2 a = \sec^2 a - 1,$

$tg\alpha = \pm\sqrt{\sec^2\alpha - 1}$. Because the $\alpha$ – third quarter angle, then $tg\alpha \rangle 0$, and therefore $tg\alpha = \sqrt{\left(-\frac{25}{7}\right)^2 - 1} = \sqrt{\frac{625}{49} - 1} = \sqrt{\frac{625-49}{49}} = \sqrt{\frac{576}{49}} = \frac{24}{7}$. From formulas of reciprocals $tg\alpha \cdot ctg\alpha = 1$ and $\cos\alpha \cdot \sec\alpha = 1$ find $ctg\alpha = \frac{1}{tg\alpha} = \frac{1}{\frac{24}{7}} = \frac{7}{24}$ i

$\cos\alpha = \frac{1}{\sec\alpha} = \frac{1}{-\frac{25}{7}} = -\frac{7}{25}$.

Division formulas give $\frac{\sin\alpha}{\cos\alpha} = tg\alpha$, $\sin\alpha = tg\alpha \cdot \cos\alpha = \frac{24}{7} \cdot \left(-\frac{7}{25}\right) = -\frac{24}{25}$.

Answer: $\sin\alpha = -\frac{24}{25}$; $\cos\alpha = -\frac{7}{25}$; $tg\alpha = \frac{24}{7}$; $ctg\alpha = \frac{7}{24}$; $\operatorname{cosec}\alpha = -\frac{25}{24}$.

It is advisable to do this also in the case when all necessary trigonometric functions are expressed through one of them. This work is necessary when solving trigonometric equations, inequalities, bringing identities, etc..

Prove identity: $2 \cdot (\cos^6\alpha + \sin^6\alpha) - 3 \cdot (\cos^4\alpha + \sin^4\alpha) = -1$.

Solution:

Recommendation: The difficult part (ie, right) should be simplified:

L.S. $= 2 \cdot (\cos^6\alpha + \sin^6\alpha) - 3 \cdot (\cos^4 + \sin^4\alpha) = 2 \cdot (\cos^4\alpha - \cos^2\alpha \cdot \sin^2\alpha + \sin^4\alpha) -$

| | |
|---|---|
| $\cos^6\alpha + \sin^6\alpha = (\cos^2\alpha)^3 + (\sin^2)^3 =$ <br> $= (\cos^2\alpha + \sin^2\alpha) \cdot (\cos^4\alpha - \cos^2\alpha \cdot \sin^2\alpha + \sin^4\alpha) =$ <br> $= 1 \cdot (\cos^4\alpha - \cos^2\alpha \cdot \sin^2\alpha + \sin^4\alpha)$. | $-3(\cos^4\alpha + \sin^4\alpha) =$ <br> $2\cos^4\alpha - 2\cos^2\alpha \cdot \sin^2\alpha +$ <br> $+ 2\sin^4\alpha - 3\cos^4\alpha - 3\sin^4\alpha =$ |

$$= -\cos^4\alpha - 2\cos^2\alpha \cdot \sin^2\alpha - \sin^4\alpha = -(\cos^4\alpha + 2\cos^2 \cdot \sin^2\alpha + \sin^4\alpha) =$$
$$= -(\cos^2\alpha + \sin^2\alpha)^2 = -1^2 = -1 = п.ч.$$

L.S. = R.S.

If the left and right sides of the equality are of approximately the same complexity, then it is easier to consider the part that contains the sine and cosine.

Prove identity: $\frac{1}{4\sin^2\alpha \cdot \cos^2\alpha} - 1 = \frac{(1 - tg^2\alpha)^2}{4tg^2\alpha}$.

Proof:

The left side of the identity is simpler. Let's reduce the hard part to the simple:

$$\frac{\left(1-tg^2\alpha\right)^2}{4tg^2\alpha}=\frac{\left(1-\frac{\sin^2\alpha}{\cos^2\alpha}\right)^2}{4\cdot\frac{\sin^2\alpha}{\cos^2\alpha}}=\frac{\left(\frac{\cos^2\alpha-\sin^2\alpha}{\cos^2\alpha}\right)^2}{\frac{4\cdot\sin^2\alpha}{\cos^2\alpha}}=\frac{\left(\cos^2\alpha-\sin^2\alpha\right)^2}{4\cdot\frac{\sin^2\alpha}{\cos^2\alpha}\cdot\cos^4\alpha}=$$

$$=\frac{\cos^4\alpha-2\sin^2\alpha\cdot\cos^2\alpha+\sin^4\alpha}{4\cdot\sin^2\alpha\cdot\cos^2\alpha}=$$

R.S $=\dfrac{\cos^4\alpha-2\sin^2\alpha\cdot\cos^2\alpha+\sin^4\alpha+4\sin^2\alpha\cos^2\alpha-4\sin^2\alpha\cdot\cos^2\alpha}{4\sin^2\alpha\cdot\cos^2\alpha}=$

$$=\frac{\cos^4\alpha+2\sin^2\alpha\cdot\cos^2\alpha+\sin^4\alpha-4\sin^2\alpha\cdot\cos^2\alpha}{4\sin^2\alpha\cdot\cos^2\alpha}=$$

$$=\frac{\left(\cos^2\alpha+\sin^2\alpha\right)^2-4\sin^2\alpha\cdot\cos^2\alpha}{4\sin^2\alpha\cdot\cos^2\alpha}=$$

$$=\frac{1-4\sin^2\alpha\cdot\cos^2\alpha}{4\sin^2\alpha\cdot\cos^2\alpha}=\frac{1}{4\sin^2\alpha\cdot\cos^2\alpha}-\frac{4\sin^2\alpha\cdot\cos^2\alpha}{4\sin^2\alpha\cdot\cos^2\alpha}=\frac{1}{4\sin^2\alpha\cdot\cos^2\alpha}-1=\textit{л.ч.}$$

Prove identity: $tg^2\beta+\sin^2\alpha=tg^2\beta\cdot\cos^2\alpha+\dfrac{\sin^2\alpha}{\cos^2\beta}.$

Proof:

The right side is harder, let's simplify it:

$$tg^2\beta\cdot\cos^2\alpha+\frac{\sin^2\alpha}{\cos^2\beta}=tg^2\beta\cdot\cos^2\alpha+\sin^2\alpha\cdot\frac{1}{\cos^2\beta}=tg^2\beta\cdot\cos^2\alpha+\sin^2\alpha\cdot\left(1+tg^2\beta\right)=$$

R.S.= $=tg^2\beta\cdot\left(1-\sin^2\alpha\right)+\sin^2\alpha\cdot\left(1+tg^2\beta\right)=tg^2\beta-tg^2\alpha\cdot\sin^2\alpha+\sin^2\alpha+tg^2\beta\cdot\sin^2\alpha=$

$$=tg^2\beta+\sin^2\alpha=\textit{л.ч.}$$

Then, R.S. = L.S.

Simplify: $\dfrac{\sin^2\alpha-\cos^2\beta}{\cos^2\alpha\cdot\cos^2\beta}-tg^2\alpha\cdot tg^2\beta.$

Solution:

$$\frac{\sin^2\alpha-\cos^2\beta}{\cos^2\alpha\cdot\cos^2\beta}-tg^2\alpha\cdot tg^2\beta=\frac{\sin^2\alpha}{\cos^2\alpha\cdot\cos^2\beta}-\frac{\cos^2\beta}{\cos^2\alpha\cdot\cos^2\beta}-tg^2\alpha\cdot tg^2\beta=$$

$$=tg^2\alpha\cdot\frac{1}{\cos^2\beta}-\frac{1}{\cos^2\alpha}-tg^2\alpha\cdot tg^2\beta=tg^2\alpha\cdot\left(1+tg^2\beta\right)-\left(1+tg^2\alpha\right)=tg^2\alpha\cdot tg^2\beta=$$

$$=tg^2\alpha+tg^2\beta\cdot tg^2\alpha-1-tg^2\alpha-tg^2\alpha\cdot tg^2\beta=-1.$$

Answer: –1.

Simplify expression: $\sqrt{\dfrac{1-\sin\alpha}{1+\sin\alpha}}-\sqrt{\dfrac{1+\sin\alpha}{1-\sin\alpha}}$ at $\dfrac{3\pi}{2}\langle\alpha\langle 2\pi.$

Solution:

$$\sqrt{\frac{1-\sin\alpha}{1+\sin\alpha}}-\sqrt{\frac{1+\sin\alpha}{1-\sin\alpha}}=\sqrt{\frac{(1-\sin\alpha)\cdot(1-\sin\alpha)}{(1+\sin\alpha)\cdot(1-\sin\alpha)}}-\sqrt{\frac{(1+\sin\alpha)\cdot(1+\sin\alpha)}{(1-\sin\alpha)\cdot(1+\sin\alpha)}}=$$

$$=\sqrt{\frac{(1-\sin\alpha)^2}{1-\sin^2\alpha}}-\sqrt{\frac{(1+\sin\alpha)^2}{\cos^2\alpha}}=\frac{|1-\sin\alpha|}{|\cos\alpha|}-\frac{|1+\sin\alpha|}{|\cos\alpha|}=\frac{1-\sin\alpha}{\cos\alpha}-\frac{1+\sin\alpha}{\cos\alpha}=$$

Considering that $\alpha -$ fourth quarter angle, we have: $\sin\alpha\langle 0,\ \ 1-\sin\alpha\rangle 0.$

Then $|1-\sin\alpha| = 1-\sin\alpha.\ \ 1+\sin\alpha\rangle 0,$ then $|1+\sin\alpha| = 1+\sin\alpha.\quad \cos\alpha\rangle 0,\ \ |\cos\alpha| = \cos\alpha.$

$$=\frac{1-\sin\alpha-1-\sin\alpha}{\cos\alpha}= \\ =\frac{-2\sin\alpha}{\cos\alpha}=-2tg\alpha.$$

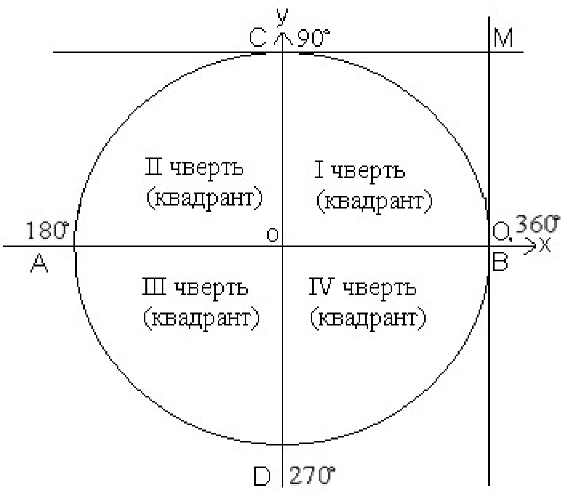
Answer: $-2tg\alpha.$

The right tools when converted trigonometric expressions is the ability to move from degree to radian measure of the argument, and vice versa. This is best done using proportion:

$$\begin{aligned}&\pi\text{рад}-180^{\circ},\\ &\alpha\text{рад}-\chi^{\circ},\end{aligned} \to \chi^{\circ}=\frac{\alpha\text{рад}\cdot 180^{\circ}}{\pi\text{рад}}.$$

$$\longrightarrow \alpha\text{рад}=\frac{\chi^{\circ}\cdot\pi\text{рад}}{180^{\circ}}.$$

It is necessary to "work out" the signs of trigonometric functions on the unit circle.




Line segment AB- cosine axis, segment CD – sinus axis, straight BM – tangent axis, straight CM – axis of cotangents. Because OB$\rangle$O "Serves" the first and fourth quarters, then the cosine of the angle is positive; OD$\langle$ O – in the third and fourth quarters, the sine is negative, etc.

The values of the trigonometric functions of some angles need to be entered into a table, which will make them easier to remember.

| $\alpha$ *в рад.* | 0 | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ | $\frac{2\pi}{3}$ | $\frac{3\pi}{4}$ | $\frac{5\pi}{6}$ | $\pi$ | $\frac{3\pi}{2}$ | $2\pi$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\sin\alpha$ | 0 | $\frac{1}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{3}}{2}$ | 1 | $\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{1}{2}$ | 0 | -1 | 0 |

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\cos\alpha$ | 1 | $\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{1}{2}$ | 0 | $-\frac{1}{2}$ | $-\frac{\sqrt{2}}{2}$ | $-\frac{\sqrt{3}}{2}$ | -1 | 0 | 1 |
| $tg\alpha$ | 0 | $\frac{1}{\sqrt{3}}$ | 1 | $\sqrt{3}$ | — | $-\sqrt{3}$ | -1 | $-\frac{1}{\sqrt{3}}$ | 0 | — | 0 |
| $ctg\alpha$ | — | $\sqrt{3}$ | 1 | $\frac{1}{\sqrt{3}}$ | 0 | $-\frac{1}{\sqrt{3}}$ | -1 | $-\sqrt{3}$ | — | 0 | — |
| $\alpha$ в *град.* | 0 | $30^\circ$ | $45^\circ$ | $60^\circ$ | $90^\circ$ | $120^\circ$ | $135^\circ$ | $150^\circ$ | $180^\circ$ | $270^\circ$ | $360^\circ$ |

This table should be supplemented with the ability to find the values of the trigonometric functions of acute angles contained in the "Four-digit tables of Bradis V.M." and the sequence of application of the "Reduction Formulas", which makes it possible to reduce the trigonometric function of an angle arbitrarily large in absolute value to the trigonometric function of an acute angle.

Mnemonic rule for using reduction formulas:

1). If the sine or cosine of an angle is converted modulo greater than $360^\circ$, then its value is divided by $360^\circ$ and work $360^\circ$ multiply by incomplete quotient omit;
In the case of tangent, or cotangent, divide by $180^\circ$ and work $180^\circ$ lower by an incomplete share.

2). The remainder of the division is served as the sum or difference of one of the limiting angles on the unit circle ($90^\circ$, $180^\circ$, $270^\circ$, $360^\circ$) and an acute angle.

3). If there is an angle in the resulting sum or difference $90^\circ$, or $270^\circ$ then the name of the function changes to similar, and if there is an angle $180^\circ$, or $360^\circ$, the function name does not change.

4). Symbol «–» or «+»result is determined by the left-hand side of the last equality in this chain of transformations.

5). In this case, it is necessary to take into account the pairing of trigonometric functions.

Let's illustrate this with examples.:

1). $\cos(-3102^\circ) = \cos 3102^\circ(\text{парність функції косинус}) = \cos\left(\underbrace{8\cdot 360^\circ}_{опускаємо} + 222^\circ\right) =$
$= \cos 222^\circ = \cos(270^\circ - 48^\circ) = -\sin 48^\circ.$

2). $tg(-2280^\circ) = -tg2280^\circ = -tg(12\cdot 180^\circ) = -tg120^\circ = -tg(90^\circ + 30^\circ) = ctg30^\circ = \sqrt{3}.$

3). $\sin\left(\alpha - 3\frac{\pi}{2}\right) = \sin\left(-\left(-\alpha + \frac{3\pi}{2}\right)\right) = \sin\left(-\left(\frac{3\pi}{2} - \alpha\right)\right) = -\sin\left(\frac{3\pi}{2} - \sin\alpha\right) = \cos\alpha.$

4) $ctg^2\left(\alpha - \frac{3\pi}{2}\right) = \left(ctg\left(\alpha - \frac{3\pi}{2}\right)\right)^2 = \left(-ctg\left(\frac{3\pi}{2} - \alpha\right)\right)^2 = (-tg\alpha)^2 = tg^2\alpha.$

# Self-study assignments:

**4.1** Calculate: $\sin 450^\circ \cdot \cos 675^\circ + tg562^\circ \cdot tg788^\circ$. Answer: $\frac{\sqrt{2}+2}{2}$.

**4.2** Prove that $\frac{1 + 2\sin\alpha\cdot\cos\alpha}{\sin^2\alpha - \cos^2\alpha} = \frac{tg\alpha + 1}{tg\alpha - 1}$.

**4.3** Prove identity: $\frac{1 - \sin\alpha}{\cos\alpha} = \frac{\cos\alpha}{1 + \sin\alpha}$.

**4.4** Prove identity: $\dfrac{1+tg\alpha}{1+ctg\alpha}=tg\alpha.$

**4.5** Prove identity: $\dfrac{(\sin\alpha+\cos\alpha)^2-1}{ctg\alpha-\sin\alpha\cdot\cos\alpha}=2tg^2\alpha.$

**4.6** Simplify expression: $\sqrt{\dfrac{1+\cos\alpha}{1-\cos\alpha}}-\sqrt{\dfrac{1-\cos\alpha}{1+\cos\alpha}}$ at $\pi\langle\alpha\langle\dfrac{3\pi}{2}$. Answer: $-2ctg\alpha.$

**4.7** Вычислить: $\dfrac{ctg\alpha-\cos\alpha}{ctg\alpha}$ at $\sin\ \alpha=\dfrac{3}{4}$. Answer: $\dfrac{1}{4}$.

**4.8** Given: $\sin\alpha+\cos\alpha=m.$ To find: а). $\sin\alpha\cdot\cos\alpha;$
б). $\sin^6\alpha+\cos^6\alpha.$

Answer: а). $\dfrac{m^2-1}{2}$; б). $\dfrac{1+6m^2-3m^4}{4}$.

**4.9** Simplify expression: $\dfrac{\sin(\pi-\alpha)-\sin3\alpha+\sin5\alpha}{2\cos2\alpha-1}$. Answer: $\sin3\alpha.$

**4.10** Simplify expression: $tg\alpha+\dfrac{\cos^3\alpha-\sin^3\alpha}{1+\sin\alpha\cdot\cos\alpha}$. Answer: 1.

**4.11** Simplify expression: $\sin^6\alpha+\cos^6\alpha+3\sin^2\alpha\cdot\cos^2\alpha$. Answer: 1.

**4.12** Calculate: $\log_{\frac{1}{2}}\left(\log_3\cos\dfrac{\pi}{6}-\log_3\sin\dfrac{\pi}{6}\right)$. Answer: 1.

**4.13** Simplify: $\dfrac{tg(180^\circ-\alpha)\cdot\cos(180^\circ-\alpha)\cdot tg(90^\circ-\alpha)}{\sin(90^\circ+\alpha)\cdot ctg(90^\circ+\alpha)\cdot tg(90^\circ+\alpha)}$. Answer: 1.

**4.14** Simplify: $\dfrac{1-2\sin^2\dfrac{\alpha}{2}}{2\cos^2\dfrac{\alpha}{2}-1}$. Answer: 1.

**4.15** Simplify: $\left(2-\dfrac{2-\cos\alpha}{1-\cos\alpha}\right)\cdot\dfrac{1-\cos\alpha}{2\cos\alpha}$. Answer: –0,5.

# Addition formulas

***This is the name of the formulas expressing trigonometric functions of angles $\alpha+\beta$ or $\alpha-\beta$ in terms of trigonometric functions of angles $\alpha$ and $\beta$ :***

$\sin(\alpha+\beta)=\sin\alpha\cdot\cos\beta+\cos\alpha\cdot\sin\beta,$ $\sin(\alpha-\beta)=\sin\alpha\cdot\cos\beta-\cos\alpha\cdot\sin\beta,$

$\cos(\alpha+\beta)=\cos\alpha\cdot\cos\beta-\sin\alpha\cdot\sin\beta,$ $\cos(\alpha-\beta)=\cos\alpha\cdot\cos\beta+\sin\alpha\cdot\sin\beta,$

$tg(\alpha+\beta)=\dfrac{tg\alpha+tg\beta}{1-tg\alpha\cdot tg\beta},$ $tg(\alpha-\beta)=\dfrac{tg\alpha-tg\beta}{1+tg\alpha\cdot tg\beta},$

$ctg(\alpha+\beta)=\dfrac{1-tg\alpha\cdot tg\beta}{tg\alpha+tg\beta},$ $ctg(\alpha-\beta)=\dfrac{1+tg\alpha\cdot tg\beta}{tg\alpha-tg\beta}.$

Let's show the application of these formulas to the solution of some exercises.
Calculate:

$$\sin105^\circ=\sin(60^\circ+45^\circ)=\sin60^\circ\cdot\cos45^\circ+\cos60^\circ\cdot\sin45^\circ=\frac{\sqrt3}{2}\cdot\frac{\sqrt2}{2}+\frac{1}{2}\cdot\frac{\sqrt2}{2}=\frac{\sqrt2}{4}\cdot(\sqrt3+1).$$

Answer: $\frac{\sqrt{2}}{4}\cdot\left(\sqrt{3}+1\right)$.

$$tg15^{\circ}=tg\left(60^{\circ}-45^{\circ}\right)=\frac{tg60^{\circ}-tg45^{\circ}}{1+tg60^{\circ}\cdot tg45^{\circ}}=\frac{\sqrt{3}-1}{1+\sqrt{3}}=\frac{\left(\sqrt{3}-1\right)\cdot\left(\sqrt{3}-1\right)}{\left(\sqrt{3}+1\right)\cdot\left(\sqrt{3}-1\right)}=\frac{\left(\sqrt{3}-1\right)^2}{3-1}=\frac{3-2\sqrt{3}+1}{2}=$$

$$=\frac{4-2\sqrt{3}}{2}=2-\sqrt{3}.$$

Answer: $2-\sqrt{3}$.

Given: $\sin\alpha=-0{,}8,\quad 180\langle\alpha\langle 270^{\circ}$ To find: $tg\left(\alpha-45^{\circ}\right)$.

Solution

$\cos^2\alpha+\sin^2\alpha=1,\quad \cos^2\alpha=1-\sin^2\alpha,\quad \cos\alpha=\pm\sqrt{1-\sin^2\alpha}$. Because the $\alpha-$ third quarter angle, then $\cos\alpha\langle 0$, and then $\cos\alpha=-\sqrt{1-\left(-0{,}8\right)^2}=-\sqrt{1-0{,}64}=-\sqrt{0{,}36}=-0{,}6$.

$$tg\alpha=\frac{\sin\alpha}{\cos\alpha}=\frac{-0{,}8}{-0{,}6}=\frac{8}{6}=\frac{4}{3}.\quad tg\left(\alpha-45^{\circ}\right)=\frac{tg\alpha-tg45^{\circ}}{1+tg\alpha\cdot tg45^{\circ}}=\frac{\frac{4}{3}-1}{1+\frac{4}{3}\cdot 1}=\frac{1\frac{1}{3}-1}{1+1\frac{1}{3}}=\frac{\frac{1}{3}}{2\frac{1}{3}}=\frac{\frac{1}{3}}{\frac{7}{3}}=\frac{1}{7}.$$

Answer: $\frac{1}{7}$.

Given: $\cos\alpha=-\frac{4}{5},\quad \pi\langle\alpha\langle\frac{3\pi}{2},\quad \sin\beta=-\frac{3}{5},\ \frac{3\pi}{2}\langle\beta\langle 2\pi.$

Calculate: $\cos\left(\alpha-\beta\right)$. Solution:

Using the basic trigonometric identity, we find $\sin\alpha=\pm\sqrt{1-\cos^2\alpha}$.

Since $\alpha-$ third quarter angle, then $\sin\alpha\langle 0$, and therefore

$$\sin\alpha=-\sqrt{1-\left(-\frac{4}{5}\right)^2}=-\sqrt{1-\frac{16}{25}}=-\sqrt{\frac{9}{25}}=-\frac{3}{5}.$$

$\cos\beta=\pm\sqrt{1-\sin^2\beta}$, $\beta-$ IV quarter angle, $\cos\beta\rangle 0$,

$$\cos\beta=\sqrt{1-\left(-\frac{3}{5}\right)^2}=\sqrt{1-\frac{9}{25}}=\sqrt{\frac{16}{25}}=\frac{4}{5}.$$

$$\cos\left(\alpha-\beta\right)=\cos\alpha\cdot\cos\beta+\sin\alpha\cdot\sin\beta=-\frac{4}{5}\cdot\frac{4}{5}-\frac{3}{5}\cdot\left(-\frac{3}{5}\right)=-\frac{16}{25}+\frac{9}{25}=-\frac{7}{25}.$$

Answer: $-\frac{7}{25}$.

Given: $\cos\left(\frac{\pi}{3}-\alpha\right)=-\frac{2}{5},\ \frac{5\pi}{6}\langle\alpha\langle\frac{4\pi}{3}$. To find $\sin\alpha$.

Solution

Let's apply this technique:

$$\sin\alpha=\left(\frac{\pi}{3}-\left(\frac{\pi}{3}-\sin\alpha\right)\right)=\sin\frac{\pi}{3}\cdot\cos\left(\frac{\pi}{3}-\alpha\right)-\cos\frac{\pi}{3}\cdot\sin\left(\frac{\pi}{3}-\alpha\right)=$$

$$=\frac{\sqrt{3}}{2}\cdot\left(-\frac{2}{5}\right)-\frac{1}{2}\cdot\sin\left(\frac{\pi}{3}-\alpha\right)=-\frac{\sqrt{3}}{5}+\frac{1}{2}\cdot\sqrt{1-\cos^2\left(\frac{\pi}{3}-\alpha\right)}=$$

Determine the sign of the expression

$$=-\frac{\sqrt{3}}{5}+\frac{1}{2}\cdot\sqrt{1-\left(-\frac{2}{5}\right)^2}=$$

$\sin\left(\frac{\pi}{3}-\alpha\right)$: by condition $\frac{5\pi}{6}\langle\alpha\langle\frac{4\pi}{3}\Big|\cdot(-1),$

$-\frac{4\pi}{3}\langle-\alpha\langle-\frac{5\pi}{6}\Big|+\frac{\pi}{3},\ \frac{\pi}{3}-\frac{4\pi}{3}\langle\frac{\pi}{3}-\alpha\langle\frac{\pi}{3}-\frac{5\pi}{6},$

$-\pi\langle\frac{\pi}{3}-\alpha\langle-\frac{3\pi}{6},\ -\pi\langle\frac{\pi}{3}-\alpha\langle-\frac{\pi}{2},$

it means that $\frac{\pi}{3}-\alpha-$ third quarter angle.

Means, $\sin\left(\frac{\pi}{3}-\alpha\right)\langle 0.$

$$=-\frac{\sqrt{3}}{5}+\frac{1}{2}\sqrt{1-\frac{4}{25}}=$$

$$=-\frac{\sqrt{3}}{5}+\frac{1}{2}\sqrt{\frac{21}{25}}=-\sqrt{\frac{3}{5}}+\sqrt{\frac{21}{10}}=$$

$$=\frac{-2\sqrt{3}+\sqrt{21}}{10}.$$

Answer: $\frac{-2\sqrt{3}+\sqrt{21}}{10}.$

Given:$\sin\alpha=\frac{2}{\sqrt{5}}$, $\cos\beta=\frac{1}{\sqrt{10}}$, $\alpha$ and $\beta-$ first quarter corners. To find $\alpha+\beta.$

Solution:

$$+\begin{array}{c}0^\circ\langle\alpha\langle 90^\circ\\ 0^\circ\langle\beta\langle 90^\circ\\ \hline 0^\circ\langle\alpha+\beta\langle 180^\circ\end{array}.$$

$$\cos(\alpha+\beta)=\cos\alpha\cdot\cos\beta-\sin\alpha\cdot\sin\beta.$$

From the main trigonometric formula we have:

$$\cos\alpha=\pm\sqrt{1-\sin^2\alpha},\ \cos\alpha=\sqrt{1-\left(\frac{2}{\sqrt{5}}\right)^2}=\sqrt{1-\frac{4}{5}}=\sqrt{\frac{1}{5}}=\frac{\sqrt{5}}{5};$$

$$\sin\beta=\pm\sqrt{1-\cos^2\beta};\quad \sin\beta=\sqrt{1-\left(\frac{1}{\sqrt{10}}\right)^2}=\sqrt{1-\frac{1}{10}}=\frac{3}{\sqrt{10}}=\frac{3\sqrt{10}}{10}.$$

Then:

$$\cos(\alpha+\beta)=\frac{\sqrt{5}}{5}\cdot\frac{1}{\sqrt{10}}-\frac{2}{\sqrt{5}}\cdot\frac{3\sqrt{10}}{10}=\frac{\sqrt{5}}{5\cdot\sqrt{5}\cdot\sqrt{2}}-\frac{6\sqrt{5}\cdot\sqrt{2}}{\sqrt{5}\cdot 10}=\frac{1}{5\sqrt{2}}-\frac{3\sqrt{2}}{5}=\frac{1-3\cdot 2}{5\sqrt{2}}=-\frac{5}{5\sqrt{2}}=-\frac{1}{\sqrt{2}}=-\frac{\sqrt{2}}{2}.$$

As $\alpha+\beta-$ angle of the II quarter, then $\alpha+\beta=180^\circ-45^\circ=135^\circ.$ Answer: $135^\circ.$

Prove that $tg\alpha\cdot tg\beta+tg\beta\cdot tg\gamma+tg\alpha\cdot tg\gamma=1,$ if $\alpha+\beta+\gamma=\frac{\pi}{2}.$

Proof:

$$tg\alpha\cdot tg\beta+tg\beta\cdot tg\gamma+tg\gamma\cdot tg\alpha=tg\alpha\cdot tg\beta+\underbrace{tg\left(\frac{\pi}{2}-(\alpha+\beta)\right)}_{tg\gamma}\cdot tg\beta+\underbrace{tg\left(\frac{\pi}{2}-(\alpha+\beta)\right)}_{tg\gamma}\cdot tg\alpha=$$

L.S.= $=tg\alpha\cdot tg\beta+tg\left(\frac{\pi}{2}-(\alpha+\beta)\right)\cdot(tg\beta+tg\alpha)=tg\alpha\cdot tg\beta+ctg(\alpha+\beta)\cdot(tg\alpha+tg\beta)=$

$$=tg\alpha\cdot tg\beta+\frac{1-tg\alpha\cdot tg\beta}{tg\alpha+tg\beta}\cdot(tg\alpha+tg\beta)=tg\alpha\cdot tg\beta+1-tg\alpha\cdot tg\beta=1=п.ч.$$

L.S. = R.S.

# Self-study assignments:

**4.16** Prove identity: $tg27^\circ+tg33^\circ+\sqrt{3}\cdot tg27^\circ\cdot tg33^\circ=3.$

**4.17** Given: $tg\alpha=\frac{1}{4},\quad 0\langle\alpha\langle\frac{\pi}{2},\ tg\beta=\frac{5}{3},\quad 0\langle\beta\langle\frac{\pi}{2}.$ To find: $\alpha-\beta.$ Answer: 0,5.

**4.18** $\cos 63^{\circ}\cdot\sin 27^{\circ}+\sin 63^{\circ}\cdot\cos 27^{\circ}$. Answer:1.

**4.19** $\cos 107^{\circ}\cdot\cos 17^{\circ}+\sin 107^{\circ}\cdot\sin 17^{\circ}$. Answer:0.

**4.20** $\cos 31^{\circ}\cdot\cos 24^{\circ}-\sin 24^{\circ}\cdot\sin 31^{\circ}-\cos 55^{\circ}$. Answer:0.

**4.21** $\sqrt{3}\cos\alpha-2\cos\left(\alpha-\frac{\pi}{6}\right)+\sin\alpha$. Answer:0.

**4.22** $\sqrt{2}\sin\left(\alpha-45^{\circ}\right)-\sin\alpha+\cos\alpha$. Answer: 0.

**4.23** $2\cos\left(60^{\circ}-\alpha\right)-\sqrt{3}\sin\alpha-\cos\alpha$. Answer: 0.

**4.24** $\sqrt{2}\sin\left(\frac{\pi}{4}+\alpha\right)-\cos\alpha-\sin\alpha$. Answer: 0.

**4.25** Given: $\sin\alpha=\frac{8}{17},\ \cos\beta=\frac{4}{5},\ \alpha$ i $\beta-$ first quarter corners.

To find: **4.26** $\sin(\alpha+\beta)$. Answer: $\frac{77}{85}$.

**4.27** $\cos(\alpha+\beta)$. Answer: $\frac{36}{85}$.

**4.28** $\cos(\alpha-\beta)$. Answer: $\frac{84}{85}$.

**4.29** $tg(\alpha+\beta)$. Answer: $2\frac{5}{36}$.

**4.30** $ctg(\alpha+\beta)$. Answer: $\frac{36}{77}$.

Prove that: **4.31** $\dfrac{tg\left(\frac{\pi}{8}+\alpha\right)+tg\left(\frac{\pi}{8}-\alpha\right)}{1-tg\left(\frac{\pi}{8}+\alpha\right)\cdot tg\left(\frac{\pi}{8}-\alpha\right)}=1.$

**4.32** $\dfrac{tg\frac{\pi}{9}+tg\frac{5\pi}{36}}{1+tg\frac{8\pi}{9}\cdot tg\frac{5\pi}{36}}=1.$

**4.33** $\dfrac{tg\alpha+tg\beta}{tg(\alpha+\beta)}+\dfrac{tg\alpha-tg\beta}{tg(\alpha-\beta)}=2.$

**4.34** $1+\mathrm{tg}\alpha\cdot tg\beta\cdot\dfrac{\cos(\alpha-\beta)}{\cos\alpha\cdot\cos\beta}=0.$

**4.35** Simplify expression: $\dfrac{2\cos^2\alpha-1}{2\sin\alpha\cdot\cos\alpha}+\dfrac{\sin 3\alpha-\sin\alpha}{\cos 3\alpha+\cos\alpha}$. Answer: $\dfrac{1}{\sin 2\alpha}$.

**4.36** Simplify an expression and compute its value:

**4.37** $\dfrac{\left(\cos^2\alpha-4\cdot\cos^2\frac{\alpha}{2}\cdot\sin^2\frac{\alpha}{2}\right)\cdot(\sin\alpha+\sin 3\alpha)}{2\cdot\cos^2\frac{\alpha}{2}-1}$ at $\alpha=\frac{\pi}{24}$. Answer: $\frac{1}{2}$.

**4.38** Prove identity: $\left(\dfrac{1}{\sin 2\alpha}-\dfrac{1}{\sin 6\alpha}\right)\cdot\dfrac{\cos 7\alpha-\cos 5\alpha}{\sin^2 2\alpha-\cos^2 2\alpha}=4\sin\alpha.$

**4.39** Calculate: $\cos(\alpha+\beta)$, if $\sin\alpha=0{,}6,\ 90^{\circ}\langle\alpha\langle 180^{\circ},\ \sin\beta=0{,}8,\ 0^{\circ}\langle\beta\langle 90^{\circ}$. Answer: 0.

# Double and triple argument formulas

***Sine of doubled argument:*** $\sin 2\alpha = 2\cdot\sin\alpha\cdot\cos\alpha.$

***Cosine of doubled argument:*** $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha,$

$$\cos 2\alpha = 2\cos^2\alpha - 1,$$

$$\cos 2\alpha = 1 - 2\sin^2\alpha.$$

***Tangent of doubled argument:*** $tg2\alpha = \dfrac{2tg\alpha}{1-tg^2\alpha},\quad tg2\alpha = \dfrac{2ctg\alpha}{ctg^2\alpha - 1}.$

***Degree reduction formulas:*** $\sin^2\alpha = \dfrac{1-\cos 2\alpha}{2},\quad \cos^2\alpha = \dfrac{1+\cos 2\alpha}{2}.$

***The sine of a triple argument:*** $\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha.$

***Cosine of a triple argument:*** $\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha.$

***The tangent of the triple argument:*** $tg3\alpha = \dfrac{3tg\alpha - tg^3\alpha}{1-3tg^2\alpha}\quad \left(\alpha \neq \dfrac{\pi}{6} + \dfrac{\pi n}{3}\right),\ n\in Z.$

***The cotangent of a triple argument:*** $ctg3\alpha = \dfrac{3ctg\alpha - ctg^3\alpha}{1-3ctg^2\alpha},\quad \left(\alpha \neq \dfrac{\pi n}{3}\right),\ n\in Z.$

We will show the application of these formulas in the process of solving exercises for applicants to universities.

Simplify expression: $\sin^2(\alpha+\beta)+\cos^2(\alpha-\beta)-\sin 2\alpha\cdot\sin 2\beta.$

Solution:

$$\sin^2(\alpha+\beta)+\cos^2(\alpha-\beta)-\sin 2\alpha\cdot\sin 2\beta = (\sin(\alpha+\beta))^2 + (\cos(\alpha-\beta))^2 - \sin 2\alpha\cdot\sin 2\beta =$$
$$= (\sin\alpha\cdot\cos\beta + \cos\alpha\cdot\sin\beta)^2 + (\cos\alpha\cdot\cos\beta + \sin\alpha\cdot\sin\beta)^2 - 2\sin\alpha\cdot\cos\alpha\cdot 2\sin\beta\cdot\cos\beta =$$
$$= \sin^2\alpha\cdot\cos^2\beta + 2\sin\alpha\cdot\cos\beta\cdot\cos\alpha\cdot\sin\beta + \cos^2\alpha\cdot\sin^2\beta + \cos^2\alpha\cdot\cos^2\beta + 2\cos\alpha\cdot\cos\beta\times$$
$$\times\sin\alpha\cdot\sin\beta + \sin^2\alpha\cdot\sin^2\beta - 4\sin\alpha\cdot\cos\alpha\cdot\sin\beta\cdot\cos\beta = \left(\sin^2\alpha\cdot\cos^2\beta + \sin^2\alpha\cdot\sin^2\beta\right) +$$
$$+ \left(\cos^2\alpha\cdot\sin^2\beta + \cos^2\alpha\cdot\cos^2\beta\right) = \sin^2\alpha\cdot\left(\cos^2\beta + \sin^2\beta\right) + \cos^2\alpha\cdot\left(\sin^2\beta + \cos^2\beta\right) =$$
$$= \sin^2\alpha + \cos^2\alpha = 1.$$

Answer: 1.

Find the value of an expression: $\cos\dfrac{\pi}{7}\cdot\cos\dfrac{2\pi}{7}\cdot\cos\dfrac{4\pi}{7}.$

Solution:

Because the $\sin\dfrac{\pi}{7}\neq 0,$ then this expression can be multiplied and divided by

$2\sin\dfrac{\pi}{7}$: $\quad \dfrac{2\sin\dfrac{\pi}{7}\cdot\cos\dfrac{\pi}{7}\cdot\cos\dfrac{2\pi}{7}\cdot\cos\dfrac{4\pi}{7}}{2\cdot\sin\dfrac{\pi}{7}} = \dfrac{2\cdot\sin\dfrac{2\pi}{7}\cdot\cos\dfrac{2\pi}{7}\cdot\cos\dfrac{4\pi}{7}}{2\cdot 2\cdot\sin\dfrac{\pi}{7}} =$

$$= \frac{2\sin\frac{4\pi}{7}\cdot\cos\frac{4\pi}{7}}{2\cdot 4\sin\frac{\pi}{7}} = \frac{\sin\frac{8\pi}{7}}{8\cdot\sin\frac{\pi}{7}} = \frac{\sin\left(\pi+\frac{\pi}{7}\right)}{8\cdot\sin\frac{\pi}{7}} = \frac{-\sin\frac{\pi}{7}}{8\cdot\sin\frac{\pi}{7}} = -\frac{1}{8}.$$

Answer: $-\dfrac{1}{8}.$

Simplify expression: $4\cos^4\alpha - 2\cos 2\alpha - \frac{1}{2}\cos 4\alpha.$

Solution:

$$4\cos^4\alpha - 2\cos 2\alpha - \frac{1}{2}\cos 4\alpha = 4(\cos^2\alpha)^2 - 2\cos 2\alpha - \frac{1}{2}\cos 4\alpha = 4\cdot\left(\frac{1+\cos 2\alpha}{2}\right)^2 - 2\cos 2\alpha - \frac{1}{2}\cos 4\alpha =$$

$$= 4\cdot\frac{1+2\cos 2\alpha+\cos^2 2\alpha}{4} - 2\cos 2\alpha - \frac{1}{2}\cos 4\alpha = 1+2\cos 2\alpha+\cos^2 2\alpha - 2\cos 2\alpha - \frac{1}{2}\cos 4\alpha =$$

$$= 1+\frac{1+\cos 4\alpha}{2} - \frac{1}{2}\cos 4\alpha = \frac{2+1+\cos 4\alpha - \cos 4\alpha}{2} = \frac{3}{2} = 1{,}5.$$

Answer: 1,5.

Calculate without tables and calculator: $\frac{1}{\sin 10^\circ} - \frac{\sqrt{3}}{\cos 10^\circ}.$

Solution:

$$\frac{1}{\sin 10^\circ} - \frac{\sqrt{3}}{\cos 10^\circ} = \frac{\cos 10^\circ - \sqrt{3}\sin 10^\circ}{\sin 10^\circ\cdot\cos 10^\circ} = \frac{2\cdot(\cos 10^\circ - \sqrt{3}\sin 10^\circ)}{2\sin 10^\circ\cdot\cos 10^\circ} = \frac{2\cdot 2\left(\frac{1}{2}\cdot\cos 10^\circ - \frac{\sqrt{3}}{2}\sin 10^\circ\right)}{\sin 20^\circ} =$$

$$= \frac{4\cdot(\sin 30^\circ\cdot\cos 10^\circ - \cos 30^\circ\cdot 10^\circ)}{\sin 20^\circ} = \frac{4\cdot\sin(30^\circ - 10^\circ)}{\sin 20^\circ} = \frac{4\cdot\sin 20^\circ}{\sin 20^\circ} = 4.$$

Answer: 4.

To find $tg\alpha$ , if $tg\left(\alpha - \frac{\pi}{4}\right) = \frac{3}{4}.$

Solution:

We transform the expression $tg\left(\alpha - \frac{\pi}{4}\right) = \frac{tg\alpha - tg\frac{\pi}{4}}{1+tg\alpha\cdot\frac{\pi}{4}} = \frac{tg\alpha - 1}{1+tg\alpha}.$

Using the condition, we have $\frac{tg\alpha - 1}{1+tg\alpha} = \frac{3}{4}.$

By the main property of proportion, we obtain the equality $4\cdot(tg\alpha - 1) = 3\cdot(1+tg\alpha);$
$4tg\alpha - 4 = 3 + 3tg\alpha;\quad 4tg\alpha - 3tg\alpha = 3+4;$
$tg\alpha = 7.$ Answer: 7.

# Self-study assignments:

Simplify expressions:

**4.40** $0{,}125 - \cos^2\alpha + \cos^4\alpha.$ Answer: $\frac{1}{8}\cos 4\alpha.$

**4.41** $\cos 36^\circ\cdot\cos 72^\circ.$ Answer: $\frac{1}{4}.$

**4.42** $1-\cos\left(\frac{\alpha}{2} - 3\pi\right) - \cos^2\frac{\alpha}{4} + \sin^2\frac{\alpha}{4}.$ Answer: 1.

**4.43** $\frac{\sin 2\alpha}{\sin\alpha}.$ Answer: $2\cos\alpha.$

**4.44** $\dfrac{\sin 2\alpha}{2\cos\alpha}$. Answer: $\sin\alpha$.

**4.45** $\dfrac{\sin 100^{o}}{\cos 50^{o}}$. Answer: $2\sin 50^{o}$.

**4.46** $\dfrac{\cos 36^{o}+\sin^{2}18^{o}}{\cos^{2}18^{o}}$. Answer: 1.

Prove that:

**4.47** $tg\alpha + 2tg2\alpha + 4ctg4\alpha = ctg\alpha$.

**4.48** $\cos\alpha\cdot\cos 2\alpha\cdot\cos 4\alpha\cdot\cos 8\alpha = \dfrac{\sin\cdot 16\alpha}{16\cdot\sin\alpha}$.

**4.49** $\sin 10^{o}\cdot\sin 50^{o}\cdot\sin 70^{o} = \dfrac{1}{8}$.

**4.50** Given: $\sin\alpha = 0{,}8,\;\; 0^{o}\langle\alpha\langle 90^{o}$.

Calculate:

**4.51** $\sin 2\alpha$. Answer: 0,96.

**4.52** $\cos 2\alpha$. Answer: – 0,28.

**4.53** $tg2\alpha$. Answer: $-\dfrac{24}{7}$.

**4.54** $ctg2\alpha$. Answer: $-\dfrac{7}{24}$.

**4.55** $\sin 15^{o}\cdot\cos 15^{o}$. Answer: $\dfrac{1}{4}$.

**4.56** $12\sin 15^{o}\cdot\sin 105^{o}$. Answer: 3.

**4.57** $6\cos 75^{o}\cdot\cos 15^{o}$. Answer: 1,5.

**4.58** $15\sin 165^{o}\cdot\cos 15^{o}$. Answer: 3,75.

**4.59** $3\sin^{2}30^{o}+8\cos^{2}30^{o}$. Answer: 6,75.

**4.60** $2\cos^{2}45^{o}+6\sin^{2}45^{o}$. Answer: $2+2\sqrt{2}$.

**4.61** $15tg157^{o}\cdot tg427^{o}$. Answer: 15.

**4.62** $15\cos 120^{o}\cdot tg315^{o}$. Answer: 7,5.

**4.63** $\sin 15^{o}\cdot\cos 75^{o}+\cos 15^{o}\cdot\sin 75^{o}$. Answer: 1.

**4.64** $\cos 75^{o}\cdot\cos 15^{o}+\sin 75^{o}\cdot\sin 15^{o}$. Answer: $\dfrac{\sqrt{3}}{2}$.

**4.65** $\dfrac{(\sin x+\cos x)^{2}-1}{\sin 2x}$. Answer: 1.

**4.66** $2\cos 4\alpha + 2\sin 4\alpha\cdot tg2\alpha$. Answer: 2.

**4.67** $\dfrac{1+\sin 2\alpha}{1+\cos 2\alpha}\cdot\dfrac{2}{(1+tg\alpha)^{2}}$. Answer: 1.

# Half Argument Trigonometric Functions

$$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}};\quad \cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}};\quad tg\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}};\quad ctg\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{1-\cos\alpha}}.$$

Given: $\sin\alpha = \dfrac{24}{25}, \quad 450^{\circ}\langle\alpha\langle 540^{\circ}.$ To find $\sin\dfrac{\alpha}{4}.$

Solution:

Finding Limits for Arguments $\dfrac{\alpha}{2}$ and $\dfrac{\alpha}{4}$: $\quad 450^{\circ}\langle\alpha\langle 540^{\circ}\Big|:2, \quad 225^{\circ}\langle\dfrac{\alpha}{2}\langle 270^{\circ}\Big|:2,$

$122{,}5^{\circ}\langle\dfrac{\alpha}{4}\langle 135^{\circ}. \quad \sin\dfrac{\alpha}{4} = \pm\sqrt{\dfrac{1-\cos\frac{\alpha}{2}}{2}}, \quad \dfrac{\alpha}{4}-$ second quarter angle, $\sin\dfrac{\alpha}{4}\rangle 0,$ then

$\sin\dfrac{\alpha}{4} = \sqrt{\dfrac{1-\cos\frac{\alpha}{2}}{2}}.$

To find $\cos\dfrac{\alpha}{2}$ find $\cos\alpha$: $\cos\alpha = \pm\sqrt{1-\sin^2\alpha}, \ \alpha -$ second quarter angle, because

$\cos\alpha\langle 0, \quad \cos\alpha = -\sqrt{1-\left(\dfrac{24}{25}\right)^2} = -\sqrt{1-\dfrac{376}{625}} = -\sqrt{\dfrac{49}{625}} = -\dfrac{7}{25}. \quad \cos\dfrac{\alpha}{2} = \pm\sqrt{\dfrac{1+\cos\alpha}{2}}. \quad \dfrac{\alpha}{2}-$

third quarter angle $\cos\dfrac{\alpha}{2}\langle 0. \quad \cos\dfrac{\alpha}{2} = -\sqrt{\dfrac{1-\frac{7}{25}}{2}} = -\sqrt{\dfrac{18}{25}\cdot\dfrac{1}{2}} = -\sqrt{\dfrac{9}{25}} = -\dfrac{3}{5}.$

$\sin\dfrac{\alpha}{4} = \sqrt{\dfrac{1+\frac{3}{5}}{2}} = \sqrt{\dfrac{8}{5}\cdot\dfrac{1}{2}} = \sqrt{\dfrac{4}{5}} = \dfrac{2}{\sqrt{5}} = \dfrac{2\cdot\sqrt{5}}{\sqrt{5}\cdot\sqrt{5}} = \dfrac{2\sqrt{5}}{5}.$ Answer: $\dfrac{2\sqrt{5}}{5}.$

Calculate: $ctg7^{\circ}30'.$ Solution:

We calculate: $\cos(2\cdot 7^{\circ}30') = \cos 15^{\circ} = \cos(60^{\circ}-45^{\circ}) = \cos 60^{\circ}\cdot\cos 45^{\circ} + \sin 60^{\circ}\cdot\sin 45^{\circ} =$

$= \dfrac{1}{2}\cdot\dfrac{\sqrt{2}}{2} + \dfrac{\sqrt{3}}{2}\cdot\dfrac{\sqrt{2}}{2} = \dfrac{\sqrt{2}+\sqrt{6}}{4},$ then $ctg7^{\circ}30' = \sqrt{\dfrac{1+\cos 15^{\circ}}{1-\cos 15^{\circ}}} = \sqrt{\dfrac{1+\frac{\sqrt{2}+\sqrt{6}}{4}}{1-\frac{\sqrt{2}+\sqrt{6}}{4}}} =$

$$= \sqrt{\dfrac{\frac{4+\sqrt{2}+\sqrt{6}}{4}}{\frac{4-\sqrt{2}-\sqrt{6}}{4}}} = \sqrt{\dfrac{4+\sqrt{2}+\sqrt{6}}{4-\sqrt{2}-\sqrt{6}}} = \sqrt{\dfrac{\sqrt{2}\cdot\left(\frac{4}{\sqrt{2}}+1+\sqrt{3}\right)}{\sqrt{2}\cdot\left(\frac{4}{\sqrt{2}}-1-\sqrt{3}\right)}} = \sqrt{\dfrac{\frac{2\cdot 2}{\sqrt{2}}+1+\sqrt{3}}{\frac{2\cdot 2}{\sqrt{2}}-1-\sqrt{3}}} =$$

$$= \sqrt{\dfrac{2\sqrt{2}+1+\sqrt{3}}{2\sqrt{2}-1-\sqrt{3}}} = \sqrt{\dfrac{\sqrt{8}+1+\sqrt{3}}{\sqrt{8}-1-\sqrt{3}}} = \sqrt{\dfrac{\sqrt{8}+\sqrt{3}+1}{\sqrt{8}-(\sqrt{3}+1)}} = \sqrt{\dfrac{(\sqrt{8}+\sqrt{3}+1)\cdot(\sqrt{8}+\sqrt{3}+1)}{(\sqrt{8}-(\sqrt{3}+1)\cdot(\sqrt{8}+\sqrt{3}+1))}} =$$

$$= \sqrt{\dfrac{(\sqrt{8}+\sqrt{3}+1)^2}{(\sqrt{8})^2-(\sqrt{3}+1)^2}} = \dfrac{|\sqrt{8}+\sqrt{3}+1|}{\sqrt{8-3-2\sqrt{3}-1}} = \dfrac{\sqrt{8}+\sqrt{3}+1}{\sqrt{4-2\sqrt{3}}} = \dfrac{\sqrt{8}+\sqrt{3}+1}{\sqrt{(\sqrt{3}-1)^2}} = \dfrac{\sqrt{8}+\sqrt{3}+1}{|\sqrt{3}-1|} =$$

$$= \dfrac{(\sqrt{8}+\sqrt{3}+1)\cdot(\sqrt{3}+1)}{(\sqrt{3}-1)\cdot(\sqrt{3}+1)} = \dfrac{\sqrt{24}+3+\sqrt{3}+\sqrt{8}+\sqrt{3}+1}{3-1} = \dfrac{2\sqrt{6}+4+2\sqrt{3}+\sqrt{8}}{2} =$$

$= \dfrac{2\sqrt{6}+2\sqrt{3}+4+2\sqrt{2}}{2} = \sqrt{6}+\sqrt{3}+\sqrt{2}+2.$ Відповідь: $\sqrt{6}+\sqrt{3}+\sqrt{2}+2.$

Simplify expression: $1+2\sin^2\alpha+\cos 2\alpha.$ Solution:

$1+2\sin^2\alpha+\cos 2\alpha=\cos^2\alpha+\sin^2\alpha+2\sin^2\alpha+\cos^2\alpha-\sin^2\alpha=$
$=2\cos^2\alpha+2\sin^2\alpha=2\cdot\left(\cos^2\alpha+\sin^2\alpha\right)=2.$ Відповідь: 2.

Calculate: $tg112^{\circ}30'$. Thus, as the angle $112^{\circ}30'-$ second quarter angle, then

$$tg112^{\circ}30'\langle 0,\ tg112^{\circ}30'=-\sqrt{\frac{1-\cos 225^{\circ}}{1+\cos 225^{\circ}}}=-\sqrt{\frac{1-\cos\left(180^{\circ}+45^{\circ}\right)}{1+\cos\left(180^{\circ}+45^{\circ}\right)}}=-\sqrt{\frac{1+\cos 45^{\circ}}{1-\cos 45^{\circ}}}=-\sqrt{\frac{1+\frac{\sqrt{2}}{2}}{1-\frac{\sqrt{2}}{2}}}=$$

$$=-\sqrt{\frac{2+\sqrt{2}}{2-\sqrt{2}}}=-\sqrt{\frac{\left(2+\sqrt{2}\right)\cdot\left(2+\sqrt{2}\right)}{\left(2-\sqrt{2}\right)\cdot\left(2+\sqrt{2}\right)}}=-\sqrt{\frac{\left(2+\sqrt{2}\right)^2}{4-2}}=-\frac{\left|2+\sqrt{2}\right|}{2}=-\frac{2+\sqrt{2}}{2}.$$

Answer: $-\frac{2+\sqrt{2}}{2}$.

Calculate: $\sin^4\frac{\pi}{16}+\sin^4\frac{3\pi}{16}+\sin^4\frac{5\pi}{16}+\sin^4\frac{7\pi}{16}$. Solution:

Using the reduction formulas, we have:

$$\sin^4\frac{\pi}{16}+\sin^4\frac{3\pi}{16}+\sin^4\frac{5\pi}{16}+\sin^4\frac{7\pi}{16}=\sin^4\frac{\pi}{16}+\sin^4\frac{3\pi}{16}+\cos^4\frac{3\pi}{16}+\cos^4\frac{\pi}{16}=$$

$$=\left(\sin^4\frac{\pi}{16}+\cos^4\frac{\pi}{16}+2\sin^2\frac{\pi}{16}\cdot\cos^2\frac{\pi}{16}-2\sin^2\frac{\pi}{16}\cdot\cos^2\frac{\pi}{16}\right)+$$

$$+\left(\sin^4\frac{3\pi}{16}+\cos^4\frac{4\pi}{16}+2\sin^2\frac{3\pi}{16}\cdot\cos^2\frac{3\pi}{16}-2\sin^2\frac{3\pi}{16}\cdot\cos^2\frac{3\pi}{16}\right)=\left(\sin^2\frac{\pi}{16}+\cos^2\frac{\pi}{16}\right)^2-$$

$$-2\sin^2\frac{\pi}{16}\cdot\cos^2\frac{\pi}{16}+\left(\sin^2\frac{3\pi}{16}+\cos^2\frac{3\pi}{16}\right)^2-2\sin^2\frac{3\pi}{16}\cdot\cos^2\frac{3\pi}{16}=1-2\sin^2\frac{\pi}{16}\cdot\cos^2\frac{\pi}{16}+$$

$$+1-2\sin^2\frac{3\pi}{16}\cdot\cos^2\frac{3\pi}{16}=2-\frac{1}{2}\left(\sin\left(2\cdot\frac{\pi}{16}\right)\right)^2-\frac{1}{2}\left(\sin\frac{\pi}{16}\left(2\cdot\frac{3\pi}{16}\right)\right)^2=2-\frac{1}{2}\sin^2\frac{\pi}{8}-\frac{1}{2}\sin^2\frac{3\pi}{8}=$$

$$=2-\frac{1}{2}\cdot\left(\sin^2\frac{\pi}{8}+\sin^2\frac{3\pi}{8}\right)=2-\frac{1}{2}\cdot\left(\sin^2\frac{\pi}{8}+\cos^2\frac{\pi}{8}\right)=2-\frac{1}{2}\cdot 1=1{,}5.$$

Answer: 1,5.

# Self-study assignments:

Calculate:

**4.68** $\sin 15^{\circ}$. Answer: $0{,}5\cdot\sqrt{2-\sqrt{3}}.$

**4.69** $\cos 15^{\circ}$. Answer: $\sqrt{2}-1$.

**4.70** $\mathrm{tg}22^{\circ}30'$. Answer: $\sqrt{2}-1$.

**4.71** $\sin 75^{\circ}$. Answer: $\frac{5}{\sqrt{26}}$.

**4.72** $\cos 75^{\circ}$. Answer: $-\frac{1}{\sqrt{26}}$.

**4.73** $tg75^{\circ}$. Answer: $-5$.

**4.74** $ctg75^{\circ}$. Answer: $-\frac{1}{5}$.

**4.75** $\cos^4\frac{\pi}{8}+\cos^4\frac{5\pi}{8}+\cos^4\frac{3\pi}{8}+\cos^4\frac{7\pi}{8}$. Answer: 1,5.

Given: $\sin\alpha=0{,}8,\ \ 90^\circ\langle\alpha\langle180^\circ$. To find:

**4.76** $\sin\frac{\alpha}{2}$. Answer: $\sqrt{0{,}8}$.

**4.77** $\cos\frac{\alpha}{2}$. Answer: $\sqrt{0{,}2}$.

**4.78** $tg\frac{\alpha}{2}$. Answer: 2.

**4.79** $ctg\frac{\alpha}{2}$. Answer: $\frac{1}{2}$.

# Expression of trigonometric functions in terms of the tangent of a half argument

$$\sin\alpha=\frac{2tg\frac{\alpha}{2}}{1+tg^2\frac{\alpha}{2}};\ \cos\alpha=\frac{1-tg^2\frac{\alpha}{2}}{1+tg^2\frac{\alpha}{2}};\ \mathrm{tg}\,\alpha=\frac{2tg\frac{\alpha}{2}}{1-tg^2\frac{\alpha}{2}};$$

Given: $\mathrm{tg}\frac{\alpha}{2}=\frac{1}{2}$. To find $x=\frac{2\sin\alpha}{4-3\cos\alpha}$.

Solution:

$$\sin\alpha=\frac{2\cdot\frac{1}{2}}{1+\left(\frac{1}{2}\right)^2}=\frac{1}{1+\frac{1}{4}}=\frac{1}{\frac{5}{4}}=\frac{4}{5};\quad \cos\alpha=\frac{1-\left(\frac{1}{2}\right)^2}{1+\left(\frac{1}{2}\right)^2}=\frac{\frac{3}{4}}{\frac{5}{4}}=\frac{3}{5},\ \text{then: } x=\frac{2\cdot\frac{4}{5}}{4-3\cdot\frac{3}{5}}=\frac{\frac{8}{5}}{4-1\frac{4}{5}}=\frac{\frac{8}{5}}{2\frac{1}{5}}=\frac{\frac{8}{5}}{\frac{11}{5}}=\frac{8}{11}.$$

Answer: $\frac{8}{11}$.

Given: $ctg\alpha=\frac{1}{3}$. To find: $\cos4\alpha$.

Solution:

$$tg\alpha=\frac{1}{ctg\alpha}=\frac{1}{\frac{1}{3}}=3;\quad tg2\alpha=\frac{2tg\alpha}{1-tg^2\alpha}=\frac{2\cdot3}{1-3^2}=-\frac{6}{8}=-\frac{3}{4};\quad \cos4\alpha=\frac{1-\left(-\frac{3}{4}\right)^2}{1+\left(-\frac{3}{4}\right)^2}=\frac{1-\frac{9}{16}}{1+\frac{9}{16}}=\frac{\frac{7}{16}}{\frac{25}{16}}=\frac{7}{25}.$$

Answer: $\frac{7}{25}$.

Given: $\mathrm{tg}2\alpha=4$. Calculate: $\sin4\alpha+\cos4\alpha\cdot ctg2\alpha$.

Solution:

$$\sin4\alpha=\frac{2tg2\alpha}{1+tg^2 2\alpha}=\frac{2\cdot4}{1+4^2}=\frac{8}{17};\quad \cos4\alpha=\frac{1-tg^2 2\alpha}{1+tg^2 2\alpha}=\frac{1-4^2}{1+4^2}=-\frac{15}{17};\quad ctg2\alpha=\frac{1}{tg2\alpha}=\frac{1}{4}.$$

Then: $\frac{8}{17}+\left(-\frac{15}{17}\right)\cdot\frac{1}{4}=\frac{8}{17}-\frac{15}{68}=\frac{32-15}{68}=\frac{17}{68}=\frac{1}{4}$. Відповідь: $\frac{1}{4}$.

# Self-study assignments:

**4.80** Given: $tg\frac{\alpha}{2}=0{,}5.$ To find $\sin^4\alpha-\cos^4\alpha.$ Answer: $\frac{7}{25}$.

To find:

**4.81** $\sin 4\alpha,$ if $tg2\alpha=3.$ Answer: 0,6.

**4.82** $\cos 4\alpha,$ if $tg2\alpha=8.$ Answer: $-\frac{63}{65}$.

**4.83** $\sin\alpha,$ if $tg\frac{\alpha}{2}=0{,}5.$ Answer: $\frac{4}{5}$.

**4.84** $\cos\alpha,$ if $tg\frac{\alpha}{2}=0{,}5.$ Answer: $\frac{3}{5}$.

**4.85** $tg\alpha,$ if $tg\frac{\alpha}{2}=0{,}5.$ Answer: $\frac{4}{3}$.

**4.86** $ctg\alpha,$ if $tg\frac{\alpha}{2}=0{,}5.$ Answer: $\frac{3}{4}$.

**4.87** $\sin\alpha+\cos\alpha$, if $tg\frac{\alpha}{2}=3.$ Answer: – 0,2.

**4.88** $\sin\alpha+\cos\alpha$, if $tg\frac{\alpha}{2}=3.$ Answer: 1,4.

# Convert the products of sines and cosines by the sum

$$\sin\alpha\cdot\cos\beta=\frac{\sin(\alpha-\beta)+\sin(\alpha+\beta)}{2}.$$

$$\cos\alpha\cdot\cos\beta=\frac{\cos(\alpha-\beta)+\cos(\alpha+\beta)}{2}.$$

$$\sin\alpha\cdot\sin\beta=\frac{\cos(\alpha-\beta)-\cos(\alpha+\beta)}{2}.$$

A lot of trigonometric exercises are solved using these formulas.

Prove identity: $\frac{1}{2\sin 10^\circ}-2\sin 70^\circ=1.$

Solution:

$$\text{L.S.}=\frac{1}{2\sin 10^\circ}-2\sin 70^\circ=\frac{1-4\cdot\sin 10^\circ\cdot\sin 70^\circ}{2\sin 10^\circ}=\frac{1-4\frac{\cos(10^\circ-70^\circ)-\cos(10^\circ+70^\circ)}{2}}{2\sin 10^\circ}=$$

$$=\frac{1-2\cdot\cos 60^\circ-\cos 80^\circ}{2\sin 10^\circ}=\frac{1-2\cdot\frac{1}{2}+2\cdot\cos 80^\circ}{2\sin 10^\circ}=\frac{2\cos 80^\circ}{2\cos 80^\circ}=1=\text{ п.ч.}$$

Identity proved.

Prove that: $\sin\alpha\cdot\sin 7\alpha-\sin 3\alpha\cdot\sin 5\alpha=-\sin 2\alpha\cdot\sin 4\alpha.$

Solution:

$$\sin\alpha\cdot\sin7\alpha-\sin3\alpha\cdot\sin5\alpha=\frac{\cos(\alpha-7\alpha)-\cos(\alpha+7\alpha)}{2}-\frac{\cos(3\alpha-5\alpha)-\cos(3\alpha+5\alpha)}{2}=$$

L.S = $=\dfrac{\cos6\alpha-\cos8\alpha}{2}-\dfrac{\cos2\alpha-\cos8\alpha}{2}=\dfrac{\cos6\alpha-\cos8\alpha-\cos2\alpha+\cos8\alpha}{2}=\dfrac{\cos6\alpha-\cos2\alpha}{2}=$

$$=\frac{-2\sin\frac{6\alpha+2\alpha}{2}\cdot\sin\frac{6\alpha-2\alpha}{2}}{2}=-\frac{\sin4\alpha\cdot\sin2\alpha}{1}=-\sin2\alpha\cdot\sin4\alpha=п.ч.$$

L.S. = R.S. *(Left side = Right side)*

Show that the magnitude of the expression:

$\cos^2\beta+\cos^2(\alpha+\beta)-2\cos\alpha\cdot\cos\beta\cdot\cos(\alpha+\beta)$ does not depend on the angle $\beta$.

Solution:

$$\cos^2\beta+\cos^2(\alpha+\beta)-2\cos\alpha\cdot\cos\beta\cdot\cos(\alpha+\beta)=\cos^2\beta+\cos^2(\alpha+\beta)-2\,\frac{\cos(\alpha+\beta)+\cos(\alpha-\beta)}{2}\times$$

$$\times\cos(\alpha+\beta)=\cos^2\beta+\cos^2(\alpha+\beta)-(\cos(\alpha+\beta)+\cos(\alpha-\beta))\cdot\cos(\alpha+\beta)=\cos^2\beta+\cos^2(\alpha+\beta)-$$

$$-\cos^2(\alpha+\beta)-\cos(\alpha-\beta)\cdot\cos(\alpha+\beta)=\cos^2\beta-\frac{\cos(\alpha-\beta-\alpha-\beta)+\cos(\alpha-\beta+\alpha+\beta)}{2}=$$

$$=\cos^2\beta-\frac{\cos(-2\beta)+\cos2\alpha}{2}=\frac{1+\cos2\beta}{2}-\frac{\cos2\beta+\cos2\alpha}{2}=\frac{1+\cos2\beta-\cos2\beta-\cos2\alpha}{2}=$$

$=\dfrac{1-\cos2\alpha}{2}=\sin^2\alpha$ Не залежить від $\beta$.

Convert to sum: $\cos^2x\cdot\cos3x$.

Solution:

$$\cos^2x\cdot\cos3x=\frac{1+\cos2x}{2}\cdot\cos3x=\frac{1}{2}\cos3x+\frac{\cos2x\cdot\cos3x}{2}=\frac{1}{2}\cdot\cos3x+\frac{1}{2}\cos2x\cdot\cos3x=$$

$$=\frac{1}{2}\cos3x+\frac{1}{2}\cdot\frac{\cos(2x-3x)+\cos(2x+3x)}{2}=\frac{1}{2}\cdot\cos3x+\frac{1}{4}\cdot(\cos x+\cos5x)=\frac{1}{2}\cdot\cos3x+\frac{1}{4}\cos x+\frac{1}{4}\cos5x.$$

Simplify: $\sin4^\circ\cdot\sin86^\circ-\cos2^\circ\cdot\sin6^\circ+0{,}5\cdot\sin4^\circ$.

Solution:

$$\sin4^\circ\cdot\sin86^\circ-\cos2^\circ\cdot\sin6^\circ+0{,}5\cdot\sin4^\circ=\sin4^\circ\cdot\cos4^\circ-\cos2^\circ\cdot\sin6^\circ+0{,}5\cdot\sin4^\circ=\frac{1}{2}\sin8^\circ-$$

$$-\frac{\sin(2^\circ+6^\circ)+\sin(2^\circ-6^\circ)}{2}+\frac{1}{2}\sin4^\circ=\frac{1}{2}\sin8^\circ-\frac{1}{2}\sin8^\circ-\frac{1}{2}\sin4^\circ+\frac{1}{2}\sin4^\circ=0.$$

Answer: 0.

$\cos\dfrac{11\pi}{26}\cdot\cos\dfrac{3\pi}{56}-\sin\dfrac{5\pi}{21}\cdot\sin\dfrac{2\pi}{21}-\dfrac{1}{2}\cos\dfrac{\pi}{4}$.

Solution:

$$\cos\frac{11\pi}{26}\cdot\cos\frac{3\pi}{56}-\sin\frac{5\pi}{21}\cdot\sin\frac{2\pi}{21}-\frac{1}{2}\cos\frac{\pi}{4}=-\frac{\cos\left(\frac{11\pi}{56}+\frac{3\pi}{56}\right)+\cos\frac{11\pi}{56}-\frac{3\pi}{56}}{2}-$$

$$-\frac{\left(\cos\frac{5\pi}{21}-\frac{2\pi}{21}\right)-\cos\left(\frac{5\pi}{21}+\frac{2\pi}{21}\right)}{2}-\frac{1}{2}\cdot\frac{\sqrt{2}}{2}=\frac{\cos\frac{\pi}{4}+\cos\frac{\pi}{7}}{2}-\frac{\cos\frac{\pi}{7}-\cos\frac{\pi}{3}}{2}-\frac{\sqrt{2}}{4}=$$

$$=\frac{\frac{\sqrt{2}}{2}+\cos\frac{\pi}{7}-\cos\frac{\pi}{7}+\frac{1}{2}}{2}-\frac{\sqrt{2}}{4}=\frac{\frac{\sqrt{2}+1}{2}}{2}-\frac{\sqrt{2}}{4}=\frac{\sqrt{2}+1}{4}-\frac{\sqrt{2}}{4}=\frac{1}{4}.$$

Answer: $\frac{1}{4}$.

Calculate: $\sin 20^\circ\cdot\sin 40^\circ\cdot\sin 60^\circ\cdot\sin 80^\circ$.

Solution:

We calculate the product:

$$\sin 20^\circ\cdot\sin 40^\circ=\frac{\cos(20^\circ-40^\circ)-\cos(20^\circ+40^\circ)}{2}=\frac{\cos 20^\circ-\cos 60^\circ}{2}=\frac{1}{2}\left(\cos 20^\circ-\frac{1}{2}\right)=\frac{1}{2}\cdot\cos 20^\circ-\frac{1}{4}.$$

Then:

$$\left(\frac{1}{2}\cos 20^\circ-\frac{1}{4}\right)\cdot\frac{\sqrt{3}}{2}\cdot\sin 80^\circ=\frac{\sqrt{3}}{4}\cdot\cos 20^\circ\cdot\sin 80^\circ-\frac{\sqrt{3}}{8}\sin 80^\circ=\frac{\sqrt{3}}{4}\cdot\left(\sin 80^\circ\cdot\cos 20^\circ-\frac{1}{2}\sin 80^\circ\right)=$$

$$=\frac{\sqrt{3}}{4}\cdot\left(\frac{\sin(80^\circ+20^\circ)+\sin(80^\circ-20^\circ)}{2}-\frac{1}{2}\cdot\sin 80^\circ\right)=\frac{\sqrt{3}}{4}\cdot\left(\frac{1}{2}\sin 100^\circ+\sin 60^\circ-\frac{1}{2}\sin 80^\circ\right)=$$

$$=\frac{\sqrt{3}}{8}\cdot\left(\sin 100^\circ+\frac{\sqrt{3}}{2}-\sin 80^\circ\right)=\frac{\sqrt{3}}{8}\cdot\left(\sin(90^\circ+10^\circ)+\frac{\sqrt{3}}{2}-\sin(90^\circ-10^\circ)\right)=\frac{\sqrt{3}}{8}\cdot\left(\cos 10^\circ+\frac{\sqrt{3}}{2}-\cos 10^\circ\right)=$$

$=\frac{3}{16}$. Відповідь $\frac{3}{16}$.

# Self-study assignments:

**4.89** Prove identity: $4\sin\alpha\cdot\sin(60^\circ-\alpha)\cdot\sin(60^\circ+\alpha)=\sin 3\alpha$.

Calculate:

**4.90** $\cos 5^\circ\cdot\cos 55^\circ\cdot\cos 65^\circ$. Answer: $\frac{1}{16}\cdot(\sqrt{6}+\sqrt{2})$.

**4.91** $tg20^\circ\cdot tg40^\circ\cdot tg60^\circ\cdot tg80^\circ$. Answer: 3.

**4.92** $2\cos 20^\circ\cdot\cos 40^\circ-\cos 20^\circ$. Answer: 0,5.

**4.93** $\sin 2x+2\sin(75^\circ-x)\cdot\cos(75^\circ+x)$. Answer: 0,5.

**4.94** $\cos^2 5^\circ+\cos^2 1^\circ-\cos 6^\circ\cdot\cos 4^\circ$. Answer: 1.

**4.95** $\cos 20^\circ\cdot\sin 50^\circ\cdot\cos 80^\circ$. Answer: 0,125.

Convert to sum:

**4.96** $\sin 10^\circ\cdot\cos 8^\circ\cdot\cos 6^\circ$. Answer: $\frac{1}{4}(\sin 24^\circ+\sin 12^\circ+\sin 8^\circ-\sin 4^\circ)$.

**4.97** $\cos 3x\cdot\cos 5x\cdot\cos 7$. Answer: $\frac{1}{2}(\cos 15x+\cos 5x+\cos 9x+\cos x)$.

**4.98** $\sin x\cdot\sin 2x\cdot\sin 3x\cdot\sin 4x.$ Answer: $\frac{1}{8}(1+\cos 10x-\cos 8x-\cos 6x).$

**4.99** $8\sin^3 x\cdot\cos x.$ Answer: $2\sin 2x-\sin 4x.$

# Conversion of the sum and difference of the trigonometric functions of the same name

$\sin\alpha+\sin\beta=2\sin\frac{\alpha+\beta}{2}\cdot\cos\frac{\alpha-\beta}{2}.$ $\sin\alpha-\sin\beta=2\sin\frac{\alpha-\beta}{2}\cdot\cos\frac{\alpha+\beta}{2}.$

$\cos\alpha+\cos\beta=2\cos\frac{\alpha+\beta}{2}\cdot\cos\frac{\alpha-\beta}{2}.$ $\cos\alpha-\cos\beta=-2\sin\frac{\alpha+\beta}{2}\cdot\sin\frac{\alpha-\beta}{2}.$

$tg\alpha+tg\beta=\frac{\sin(\alpha+\beta)}{\cos\alpha\cdot\cos\beta}$ $\left(\alpha\neq\frac{\pi}{2}\right),$ $\left(\beta\neq\frac{\pi}{2}\right).$ $tg\alpha-tg\beta=\frac{\sin(\alpha-\beta)}{\cos\alpha\cdot\cos\beta}$ $\left(\alpha\neq\frac{\pi}{2}\right),$ $\left(\beta\neq\frac{\pi}{2}\right).$

$ctg\alpha+ctg\beta=\frac{\sin(\alpha+\beta)}{\sin\alpha\cdot\sin\beta}$ $(\alpha\neq k\pi),$ $(\beta\neq m\pi).$ $ctg\alpha-ctg\beta=\frac{\sin(\beta-\alpha)}{\sin\alpha\cdot\sin\beta}$ $(\alpha\neq k\pi),$ $(\beta\neq m\pi).$

It is advisable to include in this group of formulas also such:

$a\cdot\sin\alpha+b\cdot\sin\beta=r\cdot\sin(\alpha+\varphi),$

where $r=\sqrt{a^2+b^2},$ $\cos\varphi=\frac{a}{\sqrt{a^2+b^2}},$ $\sin\varphi=\frac{b}{\sqrt{a^2+b^2}}.$

Convert to product, or share: $1+\sin\alpha+\cos\alpha.$

Solution:

$$1+\sin\alpha+\cos\alpha=(1+\cos\alpha)+\sin\alpha=2\cos^2\frac{\alpha}{2}+2\sin\frac{\alpha}{2}\cdot\cos\frac{\alpha}{2}=2\cos\frac{\alpha}{2}\cdot\left(\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}\right)=$$

$$=2\cos\frac{\alpha}{2}\cdot\left(\sin\left(\frac{\pi}{2}-\frac{\alpha}{2}\right)+\sin\frac{\alpha}{2}\right)=2\cos\frac{\alpha}{2}\cdot 2\sin\frac{\frac{\pi}{2}-\frac{\alpha}{2}+\frac{\alpha}{2}}{2}\cdot\cos\frac{\frac{\pi}{2}-\frac{\alpha}{2}-\frac{\alpha}{2}}{2}=2\cos\frac{\alpha}{2}\cdot 2\cdot\sin\frac{\pi}{4}\times$$

$$\times\cos\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)=4\cos\frac{\alpha}{2}\cdot\frac{\sqrt{2}}{2}\cdot\cos\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)=2\sqrt{2}\cdot\cos\frac{\alpha}{2}\cdot\cos\left(\frac{\pi}{4}-\frac{\alpha}{2}\right).$$

Answer: $2\sqrt{2}\cdot\cos\frac{\alpha}{2}\cdot\cos\left(\frac{\pi}{4}-\frac{\alpha}{2}\right).$

$$\sqrt{3}-2\cdot\sin\alpha=2\cdot\left(\frac{\sqrt{3}}{2}-\sin\alpha\right)=2\cdot(\sin 60^\circ-\sin\alpha)=2\cdot 2\cdot\sin\frac{60^\circ-\alpha}{2}\cdot\cos\frac{60^\circ+\alpha}{2}=$$

$$=4\cdot\sin\left(30^\circ-\frac{\alpha}{2}\right)\cdot\left(\cos 30^\circ+\frac{\alpha}{2}\right).$$

Answer: $4\cdot\sin\left(30^\circ-\frac{\alpha}{2}\right)\cdot\left(\cos 30^\circ+\frac{\alpha}{2}\right).$

$3\sin x+4\cos x=\sqrt{3^2+4^2}\cdot\sin(x+\varphi)=5\cdot\sin(x+\varphi),$ де $\cos\varphi=\frac{3}{5},$ $\sin\varphi=\frac{4}{5},$ $tg\varphi=\frac{\frac{4}{5}}{\frac{3}{5}}=\frac{4}{3},$

$\varphi=arctg\frac{4}{3}.$ Answer: $5\sin+\left(arctg\frac{4}{3}\right).$

$$\frac{\cos\alpha+\sqrt{3}\cdot\sin\alpha}{\cos\alpha-\sqrt{3}\cdot\sin\alpha}=\frac{2\left(\frac{1}{2}\cdot\cos\alpha+\frac{\sqrt{3}}{2}\cdot\sin\alpha\right)}{2\left(\frac{1}{2}\cdot\cos\alpha-\frac{\sqrt{3}}{2}\cdot\sin\alpha\right)}=\frac{\cos60^{\circ}\cdot\cos\alpha+\sin60^{\circ}\cdot\sin\alpha}{\cos60^{\circ}\cdot\cos\alpha-\sin60^{\circ}\cdot\sin\alpha}=$$

$$=\frac{\cos(60^{\circ}-\alpha)}{\cos(60^{\circ}+\alpha)}.\qquad \text{Відповідь}: \frac{\cos(60^{\circ}-\alpha)}{\cos(60^{\circ}+\alpha)}.$$

Prove that $\sin A+\sin B+\sin C=4\cos\frac{A}{2}\cdot\cos\frac{B}{2}\cdot\cos\frac{C}{2}$, де $\angle A+\angle B+\angle C=180^{\circ}$.

Solution:

Because the:

C=180° – (A+B), то sinC=sin(180° – (A+B)) $=\sin(A+B)=\sin\left(2\cdot\frac{A+B}{2}\right)=2\sin\frac{A+B}{2}\cdot\cos\frac{A+B}{2}$.

L.S. =

$$\sin A+\sin B+2\sin\frac{A+B}{2}\cdot\cos\frac{A+B}{2}=2\sin\frac{A+B}{2}\cdot\cos\frac{A-B}{2}+2\sin\frac{A+B}{2}\cdot\cos\frac{A+B}{2}=$$

$$=2\sin\frac{A+B}{2}\cdot\left(\cos\frac{A-B}{2}+\cos\frac{A+B}{2}\right)=2\sin\frac{A+B}{2}\cdot2\cdot\cos\frac{\frac{A}{2}-\frac{B}{2}+\frac{A}{2}+\frac{B}{2}}{2}\cdot\cos\frac{A}{2}-\frac{B}{2}-\frac{A}{2}-\frac{B}{2}=$$

$$=4\sin\frac{A+B}{2}\cdot\cos\frac{A}{2}\cdot\cos\left(-\frac{B}{2}\right)=4\cdot\sin\frac{180^{\circ}-C}{2}\cdot\cos\frac{A}{2}\cdot\cos\frac{B}{2}=4\cdot\sin\left(90^{\circ}-\frac{C}{2}\right)\cdot\frac{A}{2}\cdot\cos\frac{B}{2}=$$

$$=4\cos\frac{A}{2}\cdot\cos\frac{B}{2}\cdot\cos\frac{C}{2}=\textit{п.ч.}\qquad \textit{л.ч.}=\textit{п.ч.}$$

Turn into a work:

$$\sin\alpha+\cos\beta=\cos\left(\frac{\pi}{2}-\alpha\right)+\cos\beta=2\cos\frac{\frac{\pi}{2}-\alpha+\beta}{2}\cdot\cos\frac{\frac{\pi}{2}-\alpha-\beta}{2}=2\cos\left(\frac{\pi}{4}-\frac{\alpha}{2}+\frac{\beta}{2}\right)\cdot\cos\left(\frac{\pi}{4}-\frac{\alpha}{2}-\frac{\beta}{2}\right).$$

In a similar way, they turn into the product of the expression:
$\sin\alpha-\cos\beta$; $\cos\alpha\pm\sin\beta$.

$$\sin\alpha+\cos\alpha=\sqrt{2}\cdot\left(\frac{1}{\sqrt{2}}\cdot\sin\alpha+\frac{1}{\sqrt{2}}\cos\alpha\right)=\sqrt{2}\cdot\left(\cos\frac{\pi}{4}\cdot\sin\alpha+\sin\frac{\pi}{4}\cdot\cos\alpha\right)=\sqrt{2}\cdot\sin\left(\alpha+\frac{\pi}{4}\right).$$

In the same way, they turn into the product of the expression: $\sin\alpha-\cos\alpha$; $\cos\alpha\pm\sin\alpha$.

Consider a way to transform such an expression into a product:

$$\sin^2\alpha-\cos^2\beta=\frac{1-\cos2\alpha}{2}-\frac{1+\cos2\beta}{2}=\frac{1-\cos2\alpha-1-\cos2\beta}{2}=-\frac{(\cos2\alpha+\cos2\beta)}{2}=$$

$$=-\frac{1}{2}\cdot2\cdot\cos\frac{2\alpha+2\beta}{2}\cdot\cos\frac{2\alpha-2\beta}{2}=-\cos(\alpha+\beta)\cdot\cos(\alpha-\beta).$$

This method is suitable for converting such expressions into a product.:
$\sin^2\alpha-\sin^2\beta$; $\cos^2\alpha-\cos^2\beta$; $\cos^2\alpha-\sin^2\beta$.

Let us reduce to a form convenient for logarithm such an expression:

$$1+\sin\alpha=1+\cos\left(\frac{\pi}{2}-\alpha\right)=2\cos^2\frac{\frac{\pi}{2}-\alpha}{2}=2\cos^2\left(\frac{\pi}{4}-\frac{\alpha}{2}\right).$$

Similarly, they turn into a product of expression: $1-\sin\alpha;\ \ \sin\alpha-1.$

Let's consider some more ways to convert trigonometric expressions to product or fraction:

$$tg\alpha+1=\frac{\sin\alpha}{\cos\alpha}+\frac{\sin\frac{\pi}{4}}{\cos\frac{\pi}{4}}=\frac{\sin\alpha\cdot\cos\frac{\pi}{4}+\cos\alpha\cdot\sin\frac{\pi}{4}}{\cos\alpha\cdot\cos\frac{\pi}{4}}=\frac{\sin\left(\alpha+\frac{\pi}{4}\right)}{\frac{\sqrt{2}}{2}\cdot\cos\alpha}=\frac{2\sin\left(\alpha+\frac{\pi}{4}\right)}{\sqrt{2}\cos\alpha}=$$

$$=\sqrt{2}\cdot\frac{\sin\left(\alpha+\frac{\pi}{4}\right)}{\cos\alpha}.$$

Similarly converted expression: $\mathrm{tg}\,\alpha-1,\ 1\pm tg\alpha$, $\mathrm{ctg}\,\alpha\pm1,\ 1\pm ctg\alpha.$

$$2\cos\alpha+1=2\left(\cos\alpha+\frac{1}{2}\right)=2\left(\cos\alpha+\frac{\pi}{3}\right)=2\cdot2\cos\frac{\alpha+\frac{\pi}{3}}{2}\cdot\cos\frac{\alpha-\frac{\pi}{3}}{2}=4\cos\left(\frac{\alpha}{2}+\frac{\pi}{6}\right)\cdot\cos\left(\frac{\alpha}{2}-\frac{\pi}{6}\right).$$

In the same way transformed expression: $1\pm2\sin\alpha$;
$\sqrt{2}\cos\alpha\pm1;\sqrt{2}\pm2\sin\alpha;2cos\alpha\pm\sqrt{3}.$

$$3\text{-}4\sin^2\alpha=3-4\frac{1-\cos2\alpha}{2}=\frac{6-4+4\cos2\alpha}{2}=1+2\cos2\alpha=2\cdot\left(\frac{1}{2}+\cos2\alpha\right)=$$

$$=2\left(\cos\frac{\pi}{3}+\cos2\alpha\right)=2\cdot2\cos\frac{\frac{\pi}{3}+2\alpha}{2}\cdot\cos\frac{\frac{\pi}{3}-2\alpha}{2}=4\cos\left(\frac{\pi}{6}+\alpha\right)\cdot\cos\left(\frac{\pi}{6}-\alpha\right).$$

In the same way transformed expression: $3-4\cos^2\alpha;\ 4\sin^2\alpha-1;\ 4\cos^2\alpha-1.$

$$3tg^2\alpha-1=3\cdot\left(tg^2\alpha-\frac{1}{3}\right)=3\cdot\left(tg^2\alpha-tg^2\frac{\pi}{6}\right)=3\cdot\left(tg\alpha-tg\frac{\pi}{6}\right)\cdot\left(tg\alpha+tg\frac{\pi}{6}\right)=$$

$$=3\cdot\frac{\sin\left(\alpha-\frac{\pi}{6}\right)}{\cos\alpha\cdot\cos\frac{\pi}{6}}\cdot\frac{\sin\left(\alpha+\frac{\pi}{6}\right)}{\cos\alpha\cdot\cos\frac{\pi}{6}}=\frac{3\cdot\sin\left(\alpha-\frac{\pi}{6}\right)\cdot\sin\left(\alpha+\frac{\pi}{6}\right)}{\cos^2\alpha\cdot\left(\frac{\sqrt{3}}{2}\right)^2}=\frac{3\cdot\sin\left(\alpha-\frac{\pi}{6}\right)\cdot\sin\left(\alpha+\frac{\pi}{6}\right)}{\frac{3}{4}\cdot\cos^2\alpha}=$$

$$=\frac{4\cdot\sin\left(\alpha-\frac{\pi}{6}\right)\cdot\sin\left(\alpha+\frac{\pi}{6}\right)}{\cos^2\alpha}.$$ Так само можна перетворити вирази :

$tg^2\alpha-3;\ \ ctg^2\alpha-3;\ \ 3ctg^2\alpha-1.$

Given: $\alpha,\ \beta,\ \gamma-$ sharp corners and $tg\alpha+tg\beta+tg\gamma=tg\alpha\cdot tg\beta\cdot tg\gamma.$

Prove that $\alpha+\beta+\gamma=180^\circ.$

Solution:

Because: $tg\alpha+tg\beta+tg\gamma=tg\alpha\cdot tg\beta\cdot tg\gamma,$ то $tg\alpha+tg\beta+tg\gamma-tg\alpha\cdot tg\beta\cdot tg\gamma=0,$

$tg\alpha+tg\beta+(tg\gamma-tg\alpha\cdot tg\beta\cdot tg\gamma)=0,\ tg\alpha+tg\beta+tg\gamma\cdot(1-tg\alpha\cdot tg\beta)=0\,|:tg\gamma\cdot(tg\alpha+tg\beta)\neq0,$

$$\frac{tg\alpha+tg\beta}{tg\gamma\cdot(tg\alpha+tg\beta)}+\frac{tg\gamma\cdot(1-tg\alpha\cdot tg\beta)}{tg\gamma\cdot(tg\alpha+tg\beta)}=0,\ \ \frac{1}{tg\gamma}+\frac{1-tg\alpha\cdot tg\beta}{tg\alpha+tg\beta}=0,\ \ ctg\gamma+ctg(\alpha+\beta)=0,$$

$$\frac{\sin(\gamma+\alpha+\beta)}{\sin\gamma\cdot\sin(\alpha+\beta)}=0\Rightarrow\begin{cases}\sin(\alpha+\beta+\gamma)=0,\\ \sin\gamma\cdot\sin(\alpha+\beta)\neq0.\end{cases}\quad \alpha+\beta+\gamma=0^\circ,\text{або }180^\circ.$$

It follows from the condition that

$$
\begin{array}{l}
\quad 0^{\circ}\langle\alpha\langle 90^{\circ},\\
+\,0^{\circ}\langle\beta\langle 90^{\circ},\\
\quad \underline{0\langle\gamma\langle 90^{\circ}.}\\
0^{\circ}\langle\alpha+\beta+\gamma\langle 270^{\circ},\\
\alpha+\beta+\gamma=180^{\circ},
\end{array}
$$

Q.E.D. ("*quod erat demonstrandum*" - thus it has been demonstrated).

Given:

$\alpha+\beta+\gamma=180^{\circ}, \sin\alpha:\sin\beta:\sin\beta\gamma=2:3:4.$ Знайти : $\cos\alpha, \cos\beta, \cos\gamma.$

Solution:

Let $k$ be the coefficient of proportionality, then:

$\sin\alpha=2k, \sin\beta=3k, \sin\gamma=4k$ .

By the condition of the task $\alpha+\beta+\gamma=\pi,$ then $\sin(\alpha+\beta)=\sin((\pi-\gamma)),$

$\sin(\alpha+\beta)=\sin\gamma.$

Similarly $\sin(\beta+\gamma)=\sin\alpha, \quad \sin(\alpha+\gamma)=\sin\beta.$

$$
\begin{array}{ll}
\sin\alpha\cdot\cos\beta+\cos\alpha\cdot\sin\beta=\sin\gamma, & 2k\cdot\cos\beta+3k\cdot\cos\alpha=4k,\\
\sin\beta\cdot\cos\gamma+\cos\beta\cdot\sin\gamma=\sin\alpha, & 3k\cdot\cos\gamma+4k\cdot\cos\beta=2k,\\
\sin\alpha\cdot\cos\gamma+\cos\alpha\cdot\sin\gamma=\sin\beta, & 4k\cdot\cos\alpha+2k\cdot\cos\gamma=3k.
\end{array}\Bigg|:k\neq 0
$$

Let us solve such a system of three equations with three variables:

$$
\begin{cases}
2\cos\beta+3\cos\alpha=4, \to \cos\alpha=\dfrac{4-2\cos\beta}{3},\\
3\cos\gamma+4\cos\beta=2,\\
4\cos\alpha+2\cos\gamma=3,
\end{cases}
\quad
\begin{cases}
3\cos\gamma+4\cos\beta=2,\\
4\cdot\dfrac{4-2\cos\beta}{3}+2\cos\gamma=3
\end{cases}
\quad
\begin{cases}
\cos\gamma=\dfrac{2-4\cos\beta}{3}\\
\dfrac{4}{3}\cdot(4-2\cos\beta)+\dfrac{2}{3}\cdot(2-4\cos\beta)=3\Big|\cdot 3.
\end{cases}
$$

$4\cdot(4-2\cos\beta)+2\cdot(2-4\cos\beta)=9, \quad 16-8\cos\beta+4-8\cos\beta=9,$

$20-16\cos\beta=9, \quad 16\cos\beta=20-9, \quad \cos\beta=\dfrac{11}{16}.$

$$
\cos\gamma=\frac{2-4\cdot\frac{11}{16}}{3}=\frac{2-\frac{11}{4}}{3}=\frac{-\frac{3}{4}}{3}=-\frac{1}{4}. \qquad \cos\alpha=\frac{4-2\cdot\frac{11}{16}}{3}=\frac{4-1\frac{3}{8}}{3}=\frac{2\frac{5}{8}}{3}=\frac{\frac{21}{8}}{3}=\frac{21}{24}.
$$

Answer: $\cos\alpha=\dfrac{7}{8}; \quad \cos\beta=\dfrac{11}{16}; \quad \cos\gamma=-\dfrac{1}{4}.$

# Self-study assignments:

**4.100** Turn into a work: $\mathrm{tg}\,\alpha+ctg\beta.$ Answer: $\dfrac{\cos(\alpha-\beta)}{\cos\alpha\cdot\sin\beta}.$

**4.101** Given: $0\langle\alpha\langle\dfrac{\pi}{2}, \quad 0\langle\beta\langle\dfrac{\pi}{2}, \quad 0\langle\gamma\langle\dfrac{\pi}{2};$

$\cos 2\alpha+\cos 2\beta+\cos 2\gamma=-1-4\cdot\cos\alpha\cdot\cos\beta\cdot\cos\gamma.$

Prove that $\alpha+\beta+\gamma=\pi$.

**4.102** Given: $\frac{5\pi}{6}\langle\alpha\langle\frac{11\pi}{6},\ \sin\left(\alpha-\frac{\pi}{3}\right)=\frac{2}{5}.$

To find: $\cos\alpha$. Answer: $\cos\alpha=-\frac{\sqrt{3}\cdot\left(\sqrt{7}+2\right)}{10}.$

Calculate:

**4.103** $\cos 20^{o}\cdot\cos 40^{o}\cdot\cos 60^{o}\cdot\cos 80^{o}.$ Answer: $\frac{1}{16}.$

**4.104** $\cos\frac{\pi}{7}\cdot\cos\frac{4\pi}{7}\cdot\cos\frac{5\pi}{7}.$ Answer: $\frac{1}{8}.$

**4.105** Given: $\frac{\sin 3\alpha}{\sin\alpha}=2{,}2.$ Prove that $\cos 2\alpha=0{,}6.$

**4.106** Prove that $ctg\frac{\alpha}{2}=\frac{1+\cos\alpha}{\sin\alpha}.$

**4.107** $ctg\frac{\alpha}{2}=\frac{\sin\alpha}{1-\cos\alpha}.$

Given: **4.108** $540^{o}\langle\alpha\langle 630^{o}.\ tg\alpha=\frac{3}{4}.$ To find $tg\frac{\alpha}{4}.$ Answer: $\frac{1-\sqrt{10}}{3}.$

**4.109** $\sin\alpha+\cos\alpha=\frac{\sqrt{7}}{2},\ \ 0\langle\alpha\langle\frac{\pi}{4}.$ To find $tg\frac{\alpha}{2}.$ Answer: $\frac{\sqrt{7}-2}{3}.$

Present as a work:

**4.110** $\sin 16^{o}+\sin 40^{o}.$ Answer: $2\cdot\sin 28^{o}\cdot\cos 12^{o}.$

**4.111** $\sin 20^{o}-\sin 40^{o}.$ Answer: $-\sqrt{3}\cdot\sin 10^{o}.$

**4.112** $\cos 20^{o}-\cos 30^{o}.$ Answer: $2\cdot\sin 25^{o}\cdot\sin 5^{o}.$

**4.113** $\cos 15^{o}+\cos 45^{o}.$ Answer: $\sqrt{3}\cdot\cos 15^{o}.$

**4.114** $tg2x+tgx.$ Answer: $\frac{\sin 3x}{\cos x\cdot\cos 2x}.$

**4.115** $tg3x-tgx.$ Answer: $\frac{\sin 2x}{\cos 3x\cdot\cos x}.$

**4.116** $\cos 18^{o}-\sin 22^{o}.$ Answer: $2\sin 25^{o}\cdot\cos 47^{o}.$

**4.117** $\cos 50^{o}+\sin 80^{o}.$ Answer: $\sqrt{3}\cdot\cos^{o}.$

**4.118** $\cos\alpha+\sin\alpha.$ Answer: $\sqrt{2}\cdot\cos\left(\alpha-45^{o}\right).$

**4.119** $tg4x+ctg2x.$ Answer: $\frac{ctg2x}{\cos 4x}.$

**4.120** $\sin^2 x-\sin^2 y.$ Answer: $\sin(x+y)\cdot\sin(x-y).$

**4.121** $1+2\cos x.$ Answer: $4\cos\left(30^{o}+0{,}5x\right)\cdot\cos 30^{o}-0{,}5x.$

**4.122** $\cos 2\alpha-\cos 4\alpha-\cos 6\alpha+\cos 8\alpha.$ Answer: $-4\cos 5x\cdot\sin 2x\cdot\sin x.$

**4.123** $3-\sqrt{3}\cdot ctg2x.$ Answer: $\frac{2\cdot\sqrt{3}\cdot\sin\left(2x-30^{o}\right)}{\sin 2x}.$

**4.124** $\frac{4\sin^2 5x-3}{4\cos^2 5x-1}.$ Answer: $-1.$

# Some information about the inverse trigonometric functions

Suppose there is a function $y = f(x)$ with scope $X$ and many meanings $Y$. If you express $x$ through $y$, that is $x = \varphi(y)$ and in this formula swap the domain of definition and the set of values, then we get the function $y = \varphi(x)$, which is called the inverse of this function $y = f(x)$.

The inverse function of the function $y = \sin x$, called arcsine, $y = \arcsin x$.

The inverse function of the function $y = \cos x$, called arccosine, $y = \arccos x$.

The inverse function of the function $y = tgx$, called arctangent, $y = arctgx$.

The inverse function of the function $y = ctgx$, called arc cotangent, $y = arcctgx$.

***Some important properties of inverse trigonometric functions:***

$\sin(\arcsin x) = x$ i $\cos(\arccos x) = x$ при $-1 \le x \le 1$;

$tg(arctgx) = x$ i $ctg(arcctgx) = x$ при $x \in R$;

$\arcsin(\sin x) = x$ при $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$;

$\arccos(\cos x) = x$ при $0 \le x \le \pi$;

$arctg(tgx) = x$ при $-\frac{\pi}{2} \langle x \langle \frac{\pi}{2}$;

$arcctg(ctgx) = x$ при $0 \langle x \pi$;

$\arcsin(-x) = -\arcsin x \quad (-1 \le x \le 1)$;

$\arccos(-x) = \pi - \arccos x \quad (-1 \le x \le 1)$;

$\arcsin(-x) = -\arcsin x \quad (-1 \le x \le 1)$;

$arctg(-x) = -arctgx \quad (-\infty \langle x \langle +\infty)$;

$arcctg(-x) = \pi - arcctgx \quad (-\infty \langle x \langle +\infty)$.

***Next, we derive such formulas:***

$\arcsin x = \arccos \sqrt{1 - x^2}$ при $0 \langle x \langle 1$;

$\arcsin x = arctg \frac{x}{\sqrt{1 - x^2}}$ при $0 \langle x \langle 1$;

$\arcsin x = arcctg \frac{\sqrt{1 - x^2}}{x}$ при $0 \langle x \langle 1$;

$\arccos x = \arcsin \sqrt{1 - x^2}$ при $0 \langle x \langle 1$;

$\arccos x = arctg \frac{\sqrt{1 - x^2}}{x} = \mathrm{arctg} \frac{\sqrt{1 - x^2}}{x}$ при $0 \langle x \langle 1$;

$\arccos x = arcctg \frac{x}{\sqrt{1 - x^2}}$ при $0 \langle x \langle 1$;

$arctgx = \arcsin \frac{x}{\sqrt{1+x^2}}$ при $0 \langle x \langle +\infty$ ;

$arctgx = \arccos \frac{1}{\sqrt{1+x^2}}$ при $0 \langle x \langle +\infty$ ;

$arctgx = arcctg \frac{1}{x}$ при $0 \langle x \langle +\infty$ ;

$arcctgx = \arcsin \frac{1}{\sqrt{1+x^2}}$ при $0 \langle x \langle +\infty$ ;

$arcctgx = \arccos \frac{x}{\sqrt{1+x^2}}$ при $0 \langle x \langle +\infty$ ;

$arcctgx = arctg \frac{1}{x}$ при $0 \langle x \langle +\infty$ ;

$\arcsin x + \arccos x = \frac{\pi}{2}$ при $-1 \le x \le 1$;

$arctgx + arcctgx = \frac{\pi}{2}$ при $x \in R$.

Let us show the application of these formulas to the solution of some exercises on the properties of inverse trigonometric functions.

Calculate: $\sin\left(\arccos\left(-\frac{1}{2}\right)\right) - arctg\left(-\frac{1}{\sqrt{3}}\right)$.

Solution:

As known $\arccos\left(-\frac{1}{2}\right) = \pi - \arccos\left(\frac{1}{2}\right)$; $\quad arctg\left(-\frac{1}{\sqrt{3}}\right) = -arctg\left(\frac{1}{\sqrt{3}}\right)$.

Then:
$$\sin\left(\arccos\left(-\frac{1}{2}\right)\right) - arctg\left(-\frac{1}{\sqrt{3}}\right) = \sin\left(\pi - \arccos\frac{1}{2}\right) + arctg\frac{1}{\sqrt{3}} = \sin\left(\pi - \frac{\pi}{3} + \frac{\pi}{6}\right) =$$
$$= \sin\frac{5\pi}{6} = \sin\left(\pi - \frac{\pi}{6}\right) = \sin\frac{\pi}{6} = \frac{1}{2}. \qquad \text{Відповідь}: \frac{1}{2}.$$

Calculate: $tg\left(\arcsin\left(-\frac{\sqrt{3}}{2}\right) + \arccos\left(-\frac{1}{2}\right) + arctg1\right)$.

Solution:

$$\arcsin\left(-\frac{\sqrt{3}}{2}\right) = -\arcsin\frac{\sqrt{3}}{2}; \quad \arccos\left(-\frac{1}{2}\right) = \pi - \arccos\frac{1}{2}; \quad arctg1 = \frac{\pi}{4};$$

$$tg\left(\arcsin\left(-\frac{\sqrt{3}}{2}\right) + \arccos\left(-\frac{1}{2}\right) + arctg1\right) = tg\left(-\frac{\pi}{3} + \pi - \frac{\pi}{3} + \frac{\pi}{4}\right) = tg\frac{-4\pi + 12\pi - 4\pi + 3\pi}{12} =$$

$$= tg\frac{7}{12}\pi = tg\left(\frac{\pi}{3} + \frac{\pi}{4}\right) = \frac{tg\frac{\pi}{3} + tg\frac{\pi}{4}}{1 - tg\frac{\pi}{3} \cdot tg\frac{\pi}{4}} = \frac{\sqrt{3}+1}{1-\sqrt{3}\cdot 1} = \frac{\left(\sqrt{3}+1\right)^2}{\left(1-\sqrt{3}\right)\left(1+\sqrt{3}\right)} = \frac{3+2\sqrt{3}+1}{1-3} = -\left(2+\sqrt{3}\right).$$

Answer: $-\left(2+\sqrt{3}\right)$.

Prove that: $\sin\left(\frac{1}{2} \cdot \arccos\frac{1}{2}\right) = \frac{1}{2}$.

Solution:

$\sin\left(\frac{1}{2}\cdot\arccos\frac{1}{2}\right)=\sin\left(\frac{1}{2}\cdot\frac{\pi}{3}\right)=\sin\frac{\pi}{6}=\frac{1}{2}.$ L.S. = R.S.

Prove that: $\sin(\arccos x)=\sqrt{1-x^2}$.

Solution:

Let the $\arccos x=\alpha$, then $x=\cos\alpha$, $0\le\alpha\le\pi$.

With basic trigonometric identities $\sin^2\alpha+\cos^2\alpha=1$, we have

$\sin^2\alpha=1-\cos^2\alpha$; $\sin\alpha=\sqrt{1-\cos^2\alpha}$;

$\sin(\arccos x)=\sqrt{1-x^2}$. Л.ч. = п.ч.

Prove: $ctg(\arccos)=\frac{x}{\sqrt{1-x^2}}$.

Solution:

Let the $\arccos x=\alpha$, then $x=\cos\alpha$. Because the $ctg\alpha=\frac{\cos\alpha}{\sin\alpha}$, and

$\sin\alpha=\sqrt{1-\cos^2\alpha}$, then

$\sin\alpha=\sqrt{1-x^2}$. In this way, $ctg(\arccos x)=\frac{x}{\sqrt{1-x^2}}$, Q.E.D.

Prove that: $\sin(arcctgx)=\frac{1}{\sqrt{1+x^2}}$.

Solution:

Let the $arcctgx=\alpha$, then $0\langle\alpha\langle\pi$, $x=ctg\alpha$.

As known $\frac{1}{\sin^2\alpha}=1+ctg^2\alpha$, $\sin^2\alpha=\frac{1}{1+ctg^2\alpha}$,

$\sin\alpha=\frac{1}{\sqrt{1+ctg^2\alpha}}$. $\sin(arcctgx)=\frac{1}{\sqrt{1+ctg^2\alpha}}=\frac{1}{\sqrt{1+x^2}}$.

Q.E.D.

Calculate: $\cos\left(\arcsin\left(-\frac{1}{3}\right)\right)$.

Solution:

$$\cos\left(\arcsin\left(-\frac{1}{3}\right)\right)=\cos\left(-\arcsin\frac{1}{3}\right)=\cos\left(\underbrace{\arcsin\frac{1}{3}}_{\alpha}\right)=\cos\alpha=\sqrt{1-\sin^2\alpha}=\sqrt{1-\left(\frac{1}{3}\right)^2}=\sqrt{1-\frac{1}{9}}=$$

$$=\sqrt{\frac{8}{9}}=\frac{2}{3}\cdot\sqrt{2}.$$ Відповідь: $\frac{2}{3}\cdot\sqrt{2}$.

Calculate: $tg\left(\arccos\left(-\frac{1}{4}\right)\right)$.

Solution:

$$tg\left(\arccos\left(-\frac{1}{4}\right)\right)=tg\left(\pi-\arccos\frac{1}{4}\right)=-tg\left(\underbrace{\arccos\frac{1}{4}}_{\alpha}\right)=-tg\alpha=-\frac{\sqrt{1-\cos^2\alpha}}{\cos\alpha}=-\frac{\sqrt{1-\frac{1}{4}^{2}}}{\frac{1}{4}}=$$

$$=-\frac{\frac{\sqrt{15}}{4}}{\frac{1}{4}}=-\sqrt{15}. \qquad \text{Відповідь}:-\sqrt{15}.$$

Calculate:

Solution:

$$\sin(arctg(-2))=\sin(\pi-arctg2)=\sin\left(\underbrace{arctg2}_{\alpha}\right)=\sin\alpha=\frac{1}{\sqrt{1+ctg^2\alpha}}=\frac{1}{\sqrt{1+2^2}}=\frac{1}{\sqrt{5}}=\frac{\sqrt{5}}{5}.$$

Answer: $\frac{\sqrt{5}}{5}$.

Calculate: $\sin\left(\arcsin\frac{5}{13}+\arcsin\frac{12}{13}\right)$.

Solution:

$$\sin\left(\underbrace{\arcsin\frac{5}{13}}_{\alpha}+\underbrace{\arcsin\frac{12}{13}}_{\beta}\right)=\sin(\alpha+\beta)=\sin\alpha\cdot\cos\beta+\cos\alpha\cdot\sin\beta=\frac{5}{13}\cdot\frac{5}{13}+\frac{12}{13}\cdot\frac{12}{13}=\frac{25+144}{169}=1$$

$$-\frac{\pi}{2}\le\alpha\le\frac{\pi}{2}, \quad \sin\alpha=\frac{5}{13}. \qquad -\frac{\pi}{2}\le\beta\le\frac{\pi}{2}, \quad \sin\beta=\frac{12}{13}.$$

$$\cos\alpha=\sqrt{1-\sin^2\alpha}=\sqrt{1-\left(\frac{5}{13}\right)^2}=\sqrt{1-\frac{25}{169}}=\frac{12}{13};$$

$$\cos\beta=\sqrt{1-\sin^2\beta}=\sqrt{1-\left(\frac{12}{13}\right)^2}=\frac{5}{13}.$$

Answer: 1.

Calculate: $tg\left(arctg\frac{1}{2}-arctg\frac{1}{4}\right)$.

Solution:

$$tg\left(\underbrace{arctg\frac{1}{2}}_{\alpha}-\underbrace{arctg\frac{1}{4}}_{\beta}\right)=\frac{tg\alpha-tg\beta}{1+tg\alpha\cdot tg\beta}=$$

$$arctg\frac{1}{2}=\alpha, \quad tg\alpha=\frac{1}{2}, \quad -\frac{\pi}{2}<\alpha<\frac{\pi}{2}.$$

$$arctg\frac{1}{4}=\beta, \quad tg\beta=\frac{1}{4}, \quad -\frac{\pi}{2}<\beta<\frac{\pi}{2}.$$

$$=\frac{\frac{1}{2}-\frac{1}{4}}{1+\frac{1}{2}\cdot\frac{1}{4}}=\frac{\frac{1}{4}}{1\frac{1}{8}}=\frac{1}{4}\cdot\frac{8}{9}=\frac{2}{9}.$$

Answer: $\frac{2}{9}$.

Calculate: $\sin\left(2\arcsin\frac{1}{7}\right)$.

Solution:

$$\sin\Bigl(2\underbrace{\arcsin\frac{1}{7}}_{\alpha}\Bigr)=\sin 2\alpha=2\sin\alpha\cdot\cos\alpha=$$

$\arcsin\frac{1}{7}=\alpha,\quad \sin\alpha=\frac{1}{7},\quad -\frac{\pi}{2}\le\alpha\le\frac{\pi}{2};$

$\cos\alpha=\sqrt{1-\sin^2\alpha}=\sqrt{1-\left(\frac{1}{7}\right)^2}=$

$=\sqrt{1-\frac{1}{49}}=\sqrt{\frac{48}{49}}=\frac{4\sqrt{3}}{7}.$

$=2\cdot\frac{1}{7}\cdot\frac{4\sqrt{3}}{7}=\frac{8\sqrt{3}}{49}.$

Answer: $\frac{8\sqrt{3}}{49}.$

Imagine $\arcsin\frac{4}{5}$ in the form of arccosine.

Solution:

Let the $\arcsin\frac{4}{5}=\alpha,$ then

$\sin\alpha=\frac{4}{5},\quad 0<\alpha<\frac{\pi}{2}.\quad \cos\alpha=\sqrt{1-\sin^2\alpha}=\sqrt{1-\left(\frac{4}{5}\right)^2}=\sqrt{1-\frac{16}{25}}=\sqrt{\frac{9}{25}}=\frac{3}{5}.$

$\alpha=\arccos\frac{3}{5},$ then $\arcsin\frac{4}{5}=\arccos\frac{3}{5}.$ Answer: $\arccos\frac{3}{5}.$

Express $\frac{\pi}{2}-\arcsin 0{,}2$ через arccosine.

Solution:

From formula $\arcsin x+\arccos x=\frac{\pi}{2}$ we have: $\arccos x=\frac{\pi}{2}-\arcsin x,$ then

$\frac{\pi}{2}-\arcsin 0{,}2=\arccos 0{,}2.$ Answer: $\arccos 0{,}2.$

Is equality correct: $\arcsin\frac{9}{41}-\arccos\frac{4}{5}=-\arcsin\frac{84}{205}$?

Solution:

$0<\arcsin\frac{9}{41}<\frac{\pi}{2};\quad 0<\arccos\frac{4}{5}<\frac{\pi}{2}\Big|\times(-1),\quad 0>-\arccos\frac{4}{5}>-\frac{\pi}{2},\quad -\frac{\pi}{2}<-\arccos\frac{4}{5}<0.$

Add term-by-term inequalities of the same content:

$$+\begin{array}{l}0<\arcsin\frac{9}{41}<\frac{\pi}{2},\\ -\frac{\pi}{2}<-\arccos\frac{4}{5}<0.\end{array}$$

$$-\frac{\pi}{2}<\arcsin\frac{9}{41}-\arccos\frac{4}{5}<\frac{\pi}{2}.$$

$0<\arcsin\frac{8}{205}<\frac{\pi}{2}\Big|\times(-1)\quad 0<-\arcsin\frac{84}{205}>-\frac{\pi}{2};\to -\frac{\pi}{2}<-\arcsin\frac{84}{205}<0.$

The left and right sides of the original equality are contained in the interval $\left[-\frac{\pi}{2};\ \frac{\pi}{2}\right].$ In this interval, the function $y=\sin x$ increases.

Calculate the sine of the left-hand side of the equality:

$$\sin\left(\underbrace{\arcsin\frac{9}{41}}_{\alpha}-\underbrace{\arccos\frac{4}{5}}_{\beta}\right)=\sin(\alpha-\beta)=\sin\alpha\cdot\cos\beta-\cos\alpha\cdot\sin\beta=$$

$$\arcsin\frac{9}{41}=\alpha,\quad \sin\alpha=\frac{9}{41},\quad 0<\alpha<\frac{\pi}{2},\quad \arccos\frac{4}{5}=\beta,\quad \cos\beta=\frac{4}{5},\quad 0<\beta<\frac{\pi}{2}.$$

$$\cos\alpha=\sqrt{1-\sin^2\alpha}=\sqrt{1-\left(\frac{9}{41}\right)^2}=\frac{40}{41};\quad \sin\beta=\sqrt{1-\cos^2\beta}=\sqrt{1-\left(\frac{4}{5}\right)^2}=\frac{3}{5};$$

$$\sin\left(-\arcsin\frac{84}{205}\right)=-\sin\left(\arcsin\frac{84}{205}\right)=-\frac{84}{205}.$$

$$=\frac{9}{41}\cdot\frac{4}{5}-\frac{40}{41}\cdot\frac{3}{5}=\frac{36}{205}-\frac{120}{205}=-\frac{84}{205}.$$

Answer: equality is true.

Present as arcsine: $arctg\frac{9}{40}+\arccos\frac{4}{5}$.

Solution:

$$0<\frac{9}{40}<1;\quad arctg0<arctg\frac{9}{40}<arctg1,\quad 0<arctg\frac{9}{40}<\frac{\pi}{4}.$$

$$\frac{\sqrt{2}}{2}<\frac{4}{5}<1,\quad \arccos 1<\arccos\frac{4}{5}<\arccos\frac{\sqrt{2}}{2},\quad 0<\arccos\frac{4}{5}<\frac{\pi}{4}.$$

Add term-by-term inequalities:

$$+\begin{array}{l}0<arctg\frac{9}{40}<\frac{\pi}{4},\\ 0<\arccos\frac{4}{5}<\frac{\pi}{4}\end{array}\qquad arctg\frac{9}{40}=\alpha,\ tg\alpha=\frac{9}{40},\ 0<\alpha<\frac{\pi}{4}.\quad \arccos\frac{4}{5}=\beta,\ \cos\beta=\frac{4}{5},\ 0<\beta<\frac{\pi}{4}.$$

$$0<arctg\frac{9}{40}+\arccos\frac{4}{5}<\frac{\pi}{2}$$

As known $\frac{1}{\sin^2\alpha}=1+ctg^2\alpha=1+\frac{1}{tg^2\alpha}=\frac{1+tg^2\alpha}{tg^2\alpha};\quad \sin^2\alpha=\frac{tg^2\alpha}{1+tg^2\alpha};$

$$\sin\alpha=\frac{tg\alpha}{\sqrt{1+tg^2\alpha}}=\frac{\frac{9}{40}}{\sqrt{1+\left(\frac{9}{40}\right)^2}}=\frac{9}{40}\cdot\frac{40}{41}=\frac{9}{41}.$$

$$\cos\alpha=\sqrt{1-\left(\frac{9}{41}\right)^2}=\sqrt{\frac{1600}{1681}}=\frac{40}{41}.$$

$$\sin(\alpha+\beta)=\sin\alpha\cdot\cos\beta+\cos\alpha\cdot\sin\beta=\frac{9}{41}\cdot\frac{4}{5}+\frac{40}{41}\cdot\frac{3}{5}=\frac{156}{205}.$$

$arctg\frac{9}{40}+\arccos\frac{4}{5}=\arcsin\frac{156}{205}$. Answer: $\arcsin\frac{156}{205}$.

What is the angle $\arcsin\frac{1}{3}+\arcsin\frac{1}{4}.$?

Solution:

$$+\begin{array}{l} 0<\arcsin\frac{1}{3}<\frac{\pi}{2} \\ 0<\arcsin\frac{3}{4}<\frac{\pi}{2} \\ \hline \end{array},$$
$$0<\arcsin\frac{1}{3}+\arcsin\frac{3}{4}<\pi.$$

In the interval $(0;\ \pi)$ function $y=\cos x$ decreases.

We calculate: $\cos\left(\underbrace{\arcsin\frac{1}{3}}_{\alpha}+\underbrace{\arcsin\frac{3}{4}}_{\beta}\right)=\cos(\alpha+\beta)=\cos\alpha\cdot\cos\beta-\sin\alpha\cdot\sin\beta=$

$\arcsin\frac{1}{3}=\alpha,\quad \sin\alpha=\frac{1}{3},\quad 0<\alpha<\frac{\pi}{2}.$

$\arcsin\frac{3}{4}=\beta,\quad \sin\beta=\frac{3}{4},\quad 0<\beta<\frac{\pi}{2}.$

$\cos\alpha=\sqrt{1-\sin^2\alpha}=\sqrt{1-\left(\frac{1}{3}\right)^2}=\sqrt{1-\frac{1}{9}}=\sqrt{\frac{8}{9}}=\frac{2\sqrt{2}}{3}.$

$\cos\beta=\sqrt{1-\left(\frac{3}{4}\right)^2}=\sqrt{1-\frac{9}{16}}=\frac{\sqrt{7}}{4}.$

$=\frac{2\sqrt{2}}{3}\cdot\frac{\sqrt{7}}{4}-\frac{1}{3}\cdot\frac{3}{4}=\frac{2\sqrt{14}-3}{12}.$

Answer: $\frac{2\sqrt{14}-3}{12}.$

Calculate: $\arcsin\left(\sin\left(-\frac{\pi}{7}\right)\right).$

Solution:

$\arcsin\left(\sin\left(-\frac{\pi}{7}\right)\right)=\arcsin\left(-\sin\frac{\pi}{7}\right)=-\arcsin\left(\sin\frac{\pi}{7}\right)=-\frac{\pi}{7}.$ Answer: $-\frac{\pi}{7}.$

Calculate: $\arccos\left(\cos\frac{6}{5}\pi\right).$

Solution:

$\arccos\left(\cos\frac{6}{5}\pi\right)=\arccos\left(\cos\left(\pi+\frac{\pi}{5}\right)\right)=\arccos\left(\cos\left(-\frac{\pi}{5}\right)\right)=\arccos\left(\cos\frac{\pi}{5}\right)=\frac{\pi}{5}.$

Answer: $\frac{\pi}{5}.$

Calculate: $arctg(tg(-3010^\circ)).$

Solution:

$arctg(tg(-3010^\circ))=arctg(-tg(16\cdot180^\circ+130^\circ))=arctg(-tg130^\circ)=-arctg(tg130^\circ)=-arctg(tg(130^\circ-180^\circ))=$
$=-arctg(tg(-50^\circ))=-arctg(-tg50^\circ)=arctg(tg50^\circ)=50^\circ.$

Answer: $50^\circ.$

# Self-study assignments:

Calculate:

**4.125** $\cos\left(arcctg(-\sqrt{3})+arctg(-\sqrt{3})+\arcsin\frac{1}{2}\right).$ Answer: $-0{,}5.$

**4.126** $\sin(\arccos 0{,}6)$. Answer: $0{,}8$.

**4.127** $tg(\arcsin x)$. Answer: $\dfrac{x}{\sqrt{1-x^2}}$.

**4.128** $\cos(arctgx)$. Answer: $\dfrac{1}{\sqrt{1+x^2}}$.

**4.129** $\cos\left(arctg\left(-\dfrac{3}{2}\right)\right)$. Answer: $\dfrac{2}{\sqrt{13}}$.

**4.130** $ctg\left(\arcsin\left(-\dfrac{1}{4}\right)\right)$. Answer: $-\sqrt{15}$.

**4.131** $\cos\left(arcctg\left(-\dfrac{7}{8}\right)\right)$. Answer: $-\dfrac{7\sqrt{113}}{113}$.

**4.132** $\cos\left(\arcsin\left(-\dfrac{12}{13}\right)+\arcsin\dfrac{4}{5}\right)$. Answer: $\dfrac{63}{65}$.

**4.133** $\cos\left(arcctg\dfrac{3}{4}+arcctg\left(-\dfrac{12}{5}\right)\right)$. Answer: $-\dfrac{16}{65}$.

**4.134** $\sin(2arctg3)$. Answer: $\dfrac{3}{5}$.

**4.135** $\sin\left(2arctg\dfrac{1}{2}-\dfrac{1}{2}\arccos\dfrac{3}{4}\right)$. Answer: $\dfrac{\sqrt{14}}{5}-\dfrac{3\sqrt{2}}{20}$.

**4.136** $\cos\left(\dfrac{1}{2}\arcsin\dfrac{4}{5}-2arcctg\left(-\dfrac{1}{2}\right)\right)$. Answer: $-\dfrac{2\sqrt{5}}{5}$.

**4.137** $tg\left(2\arcsin\dfrac{1}{3}\right)$. Answer: $\dfrac{4\sqrt{2}}{7}$.

**4.138** $\sin\left(2\arcsin\dfrac{1}{3}\right)$. Answer: $\dfrac{4\sqrt{2}}{9}$.

**4.139** $\arcsin\left(\sin\dfrac{11}{10}\pi\right)$. Answer: $-\dfrac{\pi}{10}$.

**4.140** $\arcsin\left(\cos\frac{\pi}{9}\right)$. Answer: $\frac{7\pi}{18}$.

**4.141** $\arccos\left(\sin\left(-\frac{\pi}{7}\right)\right)$. Answer: $\frac{9\pi}{14}$.

**4.142** $2arctg\frac{1}{4}+arctg\frac{7}{23}$. Answer: $\frac{\pi}{7}$.

**4.143** $\sin\left(2arctg\frac{1}{2}\right)-tg\left(\frac{1}{2}\arcsin\frac{15}{12}\right)$. Answer: $\frac{1}{5}$.

Imagine: **4.144** $\arcsin\frac{4}{5}$ as $\arccos$. Answer: $\arccos\frac{3}{5}$.

**4.145** $\arcsin\frac{12}{13}$ as $arcctg$. Answer: $arcctg\frac{5}{12}$.

**4.146** $arctg\frac{4}{3}$ as $\arcsin$. Answer: $\arcsin\frac{4}{5}$.

**4.147** $\frac{\pi}{2}-\arccos\frac{2}{3}$ through $\arcsin$. Answer: $\arcsin\frac{2}{3}$.

Prove: **4.148** $\frac{1}{3}arctg1+\frac{1}{4}\arccos\frac{1}{2}=\frac{1}{2}arctg\sqrt{3}$.

**4.149** $\arccos\frac{1}{2}+\arccos\left(-\frac{1}{7}\right)=\arccos\left(-\frac{13}{14}\right)$.

**4.150** $2arctg\frac{1}{5}+arctg\frac{1}{4}=arctg\frac{32}{43}$.