Section 3
Exponentiation. Roots. Logarithms

Degree of number *a* with natural exponent $n \rangle 1$ It is the product of *n* multipliers, each of which is equal to *a*.

$$a^n = \underbrace{a \cdot a \cdot a \cdot a}_{\text{n множників}}$$

$2^3 = 2 \cdot 2 \cdot 2 = 8$ → значення степеня.
→ степінь
→ основа степеня
→ показник степеня.

The degree of an integral number *a* with an integral real exponent L=L_0, L_1 L_2 L_3 L_4… is called the limit of the sequence of powers of the number *a* with rational exponents, which are approximate values of the number *L* up to 0,1; 0,01; 0,001… with a shortage. $a^L = \lim\limits_{n \to \infty} a^{Ln}$

By definition, we have: $a^1 = a$; $a^0 = 1$; $a^{-n} = \dfrac{1}{a^n}$; $a^{\frac{m}{n}} = \sqrt[n]{a^m}$; $\begin{pmatrix} a \notin R, \\ a \neq 0, \\ n \notin N. \end{pmatrix}$

For example: $5^1=5$; $6^0=1$; $7^{-3} = \dfrac{1}{7^3} = \dfrac{1}{343}$; $3^{\frac{3}{4}} = \sqrt[4]{3^3}$.

Properties of powers with an arbitrary real exponent:

1). $a^x \cdot a^y = a^{x+y}$ – *multiplying degrees with identical bases;*

2). $a^x : a^y = a^{x-y}$ – *divisions of degrees with the same bases;*

3). $(a^x)^y = a^{x \cdot y}$ – *exponentiation;*

4). $(a \cdot в)^x = a^x \cdot в^x$ – *raising the product to the degree;*

5). $\left(\dfrac{a}{в}\right)^x = \dfrac{a^x}{в^x}$ – *raising fractions.*

Find the value of an expression:

3.1 a). $81^{0,75} = 81^{\frac{3}{4}} = \sqrt[4]{81^3} = \sqrt[4]{(3^4)^3} = 3^3 = 27.$

b). $\left(-2\dfrac{1}{2}\right)^3 \cdot 0{,}25^2 \cdot \left((-5)^{-3}\right)^{-2} \cdot \left((0{,}1)^2\right)^{-2} = \left(-\dfrac{5}{2}\right)^3 \cdot \left(\dfrac{5}{4}\right)^2 \cdot (-5)^6 \cdot \left(\dfrac{1}{10}\right)^{-4} = -\dfrac{5^3 \cdot 1 \cdot 1}{2^3 \cdot 2^4 \cdot 5^6} \cdot (2 \cdot 5)^4 =$

$= -\dfrac{5^3 \cdot 2^4 \cdot 5^4}{2^3 \cdot 2^4 \cdot 5^6} = -\dfrac{5^7}{2^3 \cdot 5^6} = -\dfrac{5}{8}.$

c). $\left(16^{\frac{5}{4}} - 125^{\frac{2}{3}} + 9^{\frac{3}{2}} + 27^{\frac{4}{3}} + (0{,}1)^{-1}\right)^{\frac{1}{3}} = \left((2^4)^{\frac{5}{4}} - (5^3)^{\frac{2}{3}} + (3^2)^{\frac{3}{2}} + (3^3)^{\frac{4}{3}} + 10\right)^{\frac{1}{3}} =$

$= (2^5 - 5^2 + 3^3 + 3^4 + 10)^{\frac{1}{3}} = (32 - 25 + 27 + 81 + 10)^{\frac{1}{3}} = 125^{\frac{1}{3}} = (5^3)^{\frac{1}{3}} = 5.$

d).

$$\left(15\cdot 45^{\frac{5}{3}}\right):75^{1\frac{1}{3}}+3^{\frac{1}{4}}\cdot 9^{\frac{3}{8}}=\frac{15\cdot 45^{\frac{5}{3}}}{75^{1\frac{1}{3}}}+3^{\frac{1}{4}}\cdot\left(3^{2}\right)^{\frac{3}{8}}=\frac{3\cdot 5\cdot(9\cdot 5)^{\frac{5}{3}}}{\left(5^{2}\cdot 3\right)^{\frac{4}{3}}}+3^{\frac{1}{4}}\cdot 3^{\frac{3}{4}}=$$

$$=\frac{5\cdot 3\cdot\left(5\cdot 3^{2}\right)^{\frac{5}{3}}}{\left(3\cdot 5^{2}\right)^{\frac{4}{3}}}+3^{\frac{1}{4}}\cdot 3^{\frac{3}{4}}=\frac{5\cdot 5^{\frac{5}{3}}\cdot 3\cdot 3^{\frac{10}{3}}}{3^{\frac{4}{3}}\cdot 5^{\frac{8}{3}}}+3=\frac{5^{\frac{8}{3}}\cdot 3^{\frac{13}{3}}}{3^{\frac{4}{3}}\cdot 5^{\frac{8}{3}}}+3=3^{\frac{9}{3}}+3=3^{3}+3=27+3=30.$$

e).

$$\left(5^{-3}\right)^{0}+\left(81\cdot 10^{-4}\right)^{0,25}-\left(3^{-0,8}\cdot 5^{0,5}\cdot 81^{\frac{1}{5}}\right)^{-2}+\left(\frac{5}{4}\right)^{-2}=1+\left(3^{4}\right)^{0,25}\cdot\left(10^{-4}\right)^{0,25}-\left(3^{-\frac{4}{5}}\cdot 5^{\frac{1}{2}}\cdot\left(3^{4}\right)^{\frac{1}{5}}\right)^{-2}+\left(\frac{4}{5}\right)^{2}=$$

$$=1+3\cdot 10^{-1}-\left(3^{-\frac{4}{5}}\cdot 3^{\frac{4}{5}}\cdot 5^{\frac{1}{2}}\right)^{-2}+\left(\frac{4}{5}\right)^{2}=1\frac{3}{10}-\left(1\cdot 5^{-1}\right)+\frac{16}{25}=1\frac{3}{10}-\frac{1}{5}+\frac{16}{25}=1\frac{15-10+32}{50}=1\frac{37}{50}=1,74.$$

f).

$$\left(\left(a^{\frac{3}{2}}\cdot в^{-\frac{3}{2}}-a^{-\frac{3}{2}}\cdot в^{\frac{3}{2}}\right):\left(\frac{a^{2}+в^{2}}{aв}+1\right)\right)\cdot\frac{(a-в)^{-1}}{(aв)^{-\frac{1}{2}}}=\left(\left(\frac{a^{\frac{3}{2}}}{в^{\frac{3}{2}}}-\frac{в^{\frac{3}{2}}}{a^{\frac{3}{2}}}\right):\frac{a^{2}+в^{2}+aв}{aв}\right)\cdot\frac{(aв)^{\frac{1}{2}}}{a-в}=$$

$$=\frac{a^{3}-в^{3}}{(aв)^{\frac{3}{2}}}\cdot\frac{aв}{a^{2}+aв+в^{2}}\cdot\frac{(aв)^{\frac{1}{2}}}{a-в}=\frac{(a-в)\cdot\left(a^{2}+в^{2}+aв\right)\cdot(aв)^{\frac{3}{2}}}{(aв)^{\frac{3}{2}}\cdot\left(a^{2}+в^{2}+aв\right)\cdot(a-в)}=1.$$

g).

$$\left(\frac{1}{\left(a^{\frac{1}{2}}+в^{\frac{1}{2}}\right)^{-2}}-\left(\frac{\sqrt{a}-\sqrt{в}}{a^{\frac{3}{2}}-в^{\frac{3}{2}}}\right)^{-1}\right)\cdot(aв)^{-\frac{1}{2}}=\left(\left(a^{\frac{1}{2}}+в^{\frac{1}{2}}\right)^{2}-\left(\frac{a^{\frac{1}{2}}-в^{\frac{1}{2}}}{\left(a^{\frac{1}{2}}\right)^{3}-\left(в^{\frac{1}{2}}\right)^{3}}\right)^{-1}\right)\cdot\frac{1}{(aв)^{\frac{1}{2}}}=$$

$$=\left(a+2a^{\frac{1}{2}}в^{\frac{1}{2}}+в-\frac{\left(a^{\frac{1}{2}}-в^{\frac{1}{2}}\right)\cdot\left(a+2a^{\frac{1}{2}}в^{\frac{1}{2}}+в\right)}{\left(a^{\frac{1}{2}}-в^{\frac{1}{2}}\right)}\right)\cdot\frac{1}{(aв)^{\frac{1}{2}}}=\frac{(a\cdot в)^{\frac{1}{2}}}{(a\cdot в)^{\frac{1}{2}}}=1.$$

3.2 Simplify an expression and compute its value at a=1,8; $в$=0,5

$$\frac{\left(a\cdot в^{-1}+a^{-1}в+1\right)\cdot\left(a^{-1}-в^{-1}\right)}{a^{2}\cdot в^{-2}+a^{-2}\cdot в^{2}-\left(a\cdot в^{-1}+a^{-1}\cdot в\right)}=\frac{\left(\frac{a}{в}+\frac{в}{a}+1\right)\cdot\left(\frac{1}{a}-\frac{1}{в}\right)^{2}}{\frac{a^{2}}{в^{2}}+\frac{в^{2}}{a^{2}}-\left(\frac{a}{в}+\frac{в}{a}\right)}=\frac{\frac{a^{2}+в^{2}+aв}{aв}\cdot\left(\frac{в-a}{aв}\right)^{2}}{\frac{a^{4}+в^{4}}{a^{2}в^{2}}-\frac{a^{2}+в^{2}}{aв}}=$$

$$=\frac{\frac{\left(a^{2}+aв+в^{2}\right)\cdot(в-a)^{2}}{a^{3}в^{3}}}{\frac{a^{4}+в^{4}-a^{3}в^{2}-a^{2}в^{3}}{a^{2}в^{2}}}=\frac{\left(a^{2}+aв+в^{2}\right)\cdot(в-a)^{2}\cdot a^{2}в^{2}}{a^{3}в^{3}\left(a^{4}+в^{4}-a^{3}в^{2}-a^{2}в^{3}\right)}=\frac{\left(a^{2}+aв+в^{2}\right)\cdot(a-в)^{2}}{aв\cdot\left(\left(a^{4}-a^{3}в^{2}\right)-\left(a^{2}в^{3}-в^{4}\right)\right)}=$$

$$=\frac{\left(a^{2}+aв+в^{2}\right)\cdot(a-в)^{2}}{aв\cdot\left(a^{3}\cdot\left(a-в^{2}\right)-в^{3}(a-в)\right)}=\frac{\left(a^{2}+aв+в^{2}\right)\cdot(a-в)^{2}}{aв\cdot(a-в)\cdot\left(a^{3}-в^{3}\right)}=\frac{\left(a^{2}+aв+в^{2}\right)\cdot(a-в)\cdot(a-в)}{aв(a-в)(a-в)\left(a^{2}+aв+в^{2}\right)}=\frac{1}{aв}.$$

Если а=1,8; в=0,5, то $\frac{1}{aв}=\frac{1}{1,8\cdot 0,5}=\frac{1}{0,9}=\frac{10}{9}=1\frac{1}{9}$. Answer: $1\frac{1}{9}$.

Self-study assignments:

3.3 Calculate: $\left(\left(\frac{3}{4}\right)^{0}\right)^{-0,5}-7,5\cdot\left(\sqrt[4]{4^{3}}\right)^{-2-}-1+81^{0,25}-(-2)^{-4}$. Answer: 3.

Simplify expression: **3.4** $\left(\left(\frac{x}{1-x}\right)^{\frac{2}{3}}+\frac{x^{2}}{(1-x)^{\frac{5}{3}}}\right):\left((1-x)^{\frac{1}{3}}\cdot(1-x)^{-2}\right)$. Answer: x.

3.5 $\left(\left(a^{\frac{1}{3}}-x^{\frac{1}{3}}\right)^{-1}\cdot(a-x)-\frac{a+x}{a^{\frac{1}{3}}+x^{\frac{1}{3}}}\right)\cdot 2(ax)^{-\frac{1}{3}}$. Answer: 4.

3.6 $\frac{a^{3}-a}{5a-5}\cdot\frac{a-в}{a^{2}+a}$. Answer: $\frac{a-1}{5}$ at $a\neq в, a\neq 0, a\neq 1$.

3.7 $\frac{a^{2}+6a+9}{2a-4}:\frac{3a+9}{a^{2}-4}$. Answer: $\frac{a^{2}+5a+6}{2}$ at $a\neq -3, a\neq \pm 2$.

3.8 $\frac{x^{2}-y^{2}}{x^{2}+y^{2}}\cdot\frac{x^{4}-y^{4}}{x^{2}-2xy+y^{2}}$. Answer: $(x+y)^{2}$ at $x\neq y$.

3.9 $\frac{x^{2}+2xy+y^{2}}{x^{2}-y^{2}}\cdot\frac{x^{3}-y^{3}}{x^{2}+xy+y^{2}}$. Answer: $x+y$ at $x\neq \pm y$.

3.10 $\frac{\frac{1}{x+y}-\frac{1}{x-y}}{\frac{1}{x+y}+\frac{1}{x-y}}\cdot\frac{x}{y}$. Answer: -1 at $x\neq \pm y; x\neq 0; y\neq 0$.

3.11 Given: $m\rangle 0,\ n\rangle 0,\ x=a\cdot\frac{m^{3}+n^{3}}{m^{3}-n^{3}}$.

To find $y=\left((x+a)^{\frac{1}{3}}\cdot(x-a)^{-\frac{1}{3}}+(x+a)^{-\frac{1}{3}}\cdot(x-a)^{\frac{1}{3}}-2\right)^{-\frac{1}{2}}$. Answer: $\frac{\sqrt{mn}}{m-n}$.

3.12 Given: $k\rangle 0,\ x=2k^{\frac{1}{2}}\cdot(1+k)^{-1}$.

To find $y=\left(\frac{\left(1-x^{2}\right)^{-\frac{1}{2}}+1}{2}\right)^{-\frac{1}{2}}+\left(\frac{\left(1-x^{2}\right)^{-\frac{1}{2}}-1}{2}\right)^{-\frac{1}{2}}$. Answer: $\sqrt{k-1}\cdot\left(1+\frac{1}{k^{\frac{1}{2}}}\right)$.

3.13 Given: $n\rangle 1,\ x=\left(1-n^{-1}\right)^{\frac{1}{2}}\cdot\left(1+n^{-1}\right)^{-\frac{1}{2}}$.
To find $y=\left(1+x^{-1}\right)^{-2}+\left(1-x^{-1}\right)^{-2}$. Answer: $n(n-1)$.

3.14 $\left(\frac{1}{a-1}+\frac{2(a-1)}{a^{2}-4}-\frac{4(a+1)}{a^{2}+a-2}+\frac{a}{a^{2}-3a+2}\right)\cdot\frac{a^{3}-a^{2}-4a+4}{a^{3}+27}:\frac{a}{a^{2}-3a+9}$.

Answer: $\frac{2}{a}$ at $a\neq -3,\ a\neq \pm 2;\ a\neq 1,\ a\neq 0$.

3.15 $\frac{x^{2}}{x^{2}+x+1}+\frac{x}{x^{2}+x-2}+\frac{x}{x^{3}+3x^{2}+3x+2}-\frac{1}{x^{3}-1}$. Answer: 1 at $x\neq -2,\ x\neq 1$.

3.16 $\frac{(a-в)^2+ав}{(a+в)^2-ав}:\frac{a^5+в^5+a^2в^3+a^3в^2}{(a^3+в^3+a^2в+ав^2)\cdot(a^3-в^3)}$. Answer: $a-в$ at $a\neq\pm в$.

3.17 $\frac{x^3+y^3}{x+y}\cdot(x^2-y^2)^{-1}+\frac{2y}{x+y}-\frac{xy}{x^2-y^2}$. Answer: 1 at $x\neq\pm y$.

Definition and properties of a root of a number

The root of n-th degree $(n\in N,\ n\geq 2)$ number a is called the number b , n-*ой* степень которого равна a . $\sqrt[n]{a}$.

For example, $\sqrt{25}=\pm 5$. $\sqrt[3]{8}=2$, $3-$ root exponent, 8 – root expression, 2 – root value.

The definition of a root implies the identity $(\sqrt[n]{a})^n=a$, or $\sqrt[n]{a^n}=a$, $\sqrt[n]{0}$.

If n - *is an even number*, there are two values of the square root of any positive number.

For example: $\sqrt{16}=\pm 2$. If n – *odd number*, then there is only one value of the root from any real number.

For example: $\sqrt[3]{-8}=-2$; $\sqrt[3]{125}=5$.

Arithmetic root n - *degree* $(n\in N,\ n\geq 2)$ with a positive number a is called an integral number $в$, n - *th* degree is equal to a.

For example: $\sqrt[4]{81}=3$.

Finding the root of n - degree is called a root extraction.

Properties arithmetic root:

$\sqrt[n]{a\cdot в}=\sqrt[n]{a}\cdot\sqrt[n]{в}-$ the root of the product;

$\sqrt[n]{\frac{a}{в}}=\frac{\sqrt[n]{a}}{\sqrt[n]{в}}$, $(в\neq 0)-$ root of the particle;

$\sqrt[n]{\sqrt[к]{a}}=\sqrt[n\cdot к]{a}-$ root from root.

$\sqrt[n]{a}=\sqrt[nк]{a^к}$.

If $0\leq a\langle в$, then $\sqrt[n]{a}\langle\sqrt[n]{в}$.

Якщо $a\rangle 0$ і $m\in N$, т $\sqrt[2m]{a^{2m}}=|a|=\begin{cases}a, \text{ якщо } a\geq 0,\\ -a, \text{ якщо } a\langle 0.\end{cases}$

$\sqrt[2m+1]{a^{2m+1}}=a$, $\sqrt[2m+1]{-a}=-\sqrt[2m+1]{a}$. Here $2m$ – even number, *2m+1* – odd number.

We will show the application of the definition of a root of a number and its properties to solving exercises.

Find the value of an expression $\sqrt{75\cdot 48}$. Solution:

Decompose the number 75 by two factors so that at least one of them can be root $75=25\cdot 3$. Number 48 decompose into two factors so that one of them is equal to 3: $48=3\ \cdot 16$ We use the property of the root from the product:

$\sqrt{75\cdot 48}=\sqrt{25\cdot 3\cdot 3\cdot 16}=\sqrt{25}\cdot\sqrt{3\cdot 3}\cdot\sqrt{16}=5\cdot 3\cdot 4=60.$

Answer:60. $\sqrt{2\frac{3}{4}\cdot 1\frac{5}{11}}$.

Solution: turn each of the "mixed" numbers into an improper fraction, and then simplify the radical expression: $\sqrt{2\frac{3}{4}\cdot 1\frac{5}{11}} == \sqrt{\frac{11}{4}\cdot\frac{16}{11}} = \sqrt{4} = 2.$ Answer: 2.

$\sqrt[3]{0,27\cdot(-100)}$. Solution: not one of 0,27, not one of (–100) the cube root is not extracted, and therefore we transform the radical expression in this way:

$0,27\cdot(-100) = (-1)\cdot(0,27\cdot 100) = (-1)\cdot 27.$

$\sqrt[3]{0,27\cdot(-100)} = \sqrt[3]{(-1)\cdot 27} = \sqrt[3]{(-1)}\cdot\sqrt[3]{27} = (-1)\cdot 3 = -3.$ Answer: –3.

$\sqrt[6]{17^2 - 15^2}$. Solution: transform the radical expression using the formula for the difference of squares of two numbers:

$17^2 - 15^2 = (17-15)\cdot(17+15) = 2\cdot 32 = 2\cdot 2^5 = 2^6.$

$\sqrt[6]{17^2 - 15^2} = \sqrt[6]{2^6} = |2| = 2.$ Answer: 2.

$\left(\frac{12}{\sqrt{15}-3} - \frac{28}{\sqrt{15}-1} + \frac{1}{2-\sqrt{3}}\right)\cdot\left(6-\sqrt{3}\right)$. Solution:

It is advisable to first get rid of irrationality in the denominator of each of the three fractions: $\frac{12}{\sqrt{15}-3} = \frac{12\cdot(\sqrt{15}+3)}{(\sqrt{15}-3)\cdot(\sqrt{15}+3)} = \frac{12(\sqrt{15}+3)}{15-9} = \frac{12(\sqrt{15}+3)}{6} = 2\sqrt{15}+6;$

$\frac{28}{\sqrt{15}-1} = \frac{28\cdot(\sqrt{15}+1)}{(\sqrt{15}-1)\cdot(\sqrt{15}+1)} = \frac{28\cdot(\sqrt{15}+1)}{15-1} = \frac{28\cdot(\sqrt{15}+1)}{14} = 2\sqrt{15}+2;$

$\frac{1}{2-\sqrt{3}} = \frac{1\cdot(2+\sqrt{3})}{(2-\sqrt{3})\cdot(2+\sqrt{3})} = \frac{2+\sqrt{3}}{4-3} = 2+\sqrt{3}.$

$\left(2\cdot\sqrt{15}+6-2\cdot\sqrt{15}-2+2+\sqrt{3}\right)\cdot\left(6-\sqrt{3}\right) = \left(6-\sqrt{3}\right)\cdot\left(6+\sqrt{3}\right) = 36-3 = 33$. Answer: 33.

Noteworthy is this way of simplifying expressions like:

$\sqrt{6+2\sqrt{5}} - \sqrt{6-2\sqrt{5}}$. Solution: denote $\sqrt{6+2\sqrt{5}} - \sqrt{6-2\sqrt{5}} = A.$

Let us square both sides of this equality:

$6+2\cdot\sqrt{5} - 2\cdot\sqrt{6+2\cdot\sqrt{5}}\cdot\sqrt{6-2\cdot\sqrt{5}} + 6 - 2\cdot\sqrt{5} = A^2;$

$12-2\sqrt{(6+2\cdot\sqrt{5})\cdot(6-2\cdot\sqrt{5})} = A^2;$

$12-2\quad\sqrt{36-20} = A^2;$

$12-2\cdot 4 = A^2;$ $\sqrt{\sqrt{\ }}$

$A^2 = 4;$

$A = \pm 2$; Because the $\sqrt{6+2\cdot\sqrt{5}} \geq \sqrt{6-2\cdot\sqrt{5}}$, then the difference

$\sqrt{6+2\cdot\sqrt{5}} - \sqrt{6-2\cdot\sqrt{5}} \rangle 0.$

Well then, A=2. Answer: 2.

Calculate the value of expression: **3.18** $\sqrt{\frac{a\cdot\sqrt{a}-1}{\sqrt{a}-1}+\sqrt{a}} - \sqrt{\frac{a\cdot\sqrt{a}+1}{\sqrt{a}+1}-\sqrt{a}}$ at a =*2,5.*

Solution:

Let's simplify this expression by reducing the radical expressions to a common denominator. Then we use the formula for the difference of squares and after cancellation we apply the concept of the modulus of the number.

$$\sqrt{\frac{a\cdot\sqrt{a}-1}{\sqrt{a}-1}+\sqrt{a}}-\sqrt{\frac{a\cdot\sqrt{a}+1}{\sqrt{a}+1}-\sqrt{a}}=\sqrt{\frac{a\cdot\sqrt{a}-1+a-\sqrt{a}}{\sqrt{a}-1}}-\sqrt{\frac{a\cdot\sqrt{a}+1-a-\sqrt{a}}{\sqrt{a}+1}}=$$

$$=\sqrt{\frac{\sqrt{a}(a-1)+(a-1)}{\sqrt{a}-1}}-\sqrt{\frac{\sqrt{a}(a+1)-(a-1)}{\sqrt{a}+1}}=\sqrt{\frac{(a-1)(\sqrt{a}-1)}{\sqrt{a}-1}}-\sqrt{\frac{(a+1)(\sqrt{a}-1)}{\sqrt{a}+1}}=$$

$$=\sqrt{\frac{(\sqrt{a}-1)(\sqrt{a}+1)(\sqrt{a}+1)}{\sqrt{a}-1}}-\sqrt{\frac{(\sqrt{a}-1)(\sqrt{a}+1)(\sqrt{a}-1)}{\sqrt{a}+1}}=\sqrt{(\sqrt{a}+1)^2}-\sqrt{(\sqrt{a}-1)^2}=$$

$$=\left|\sqrt{a}+1\right|-\left|\sqrt{a}-1\right|.$$

If $a=2{,}5$, then $\left|\sqrt{a}+1\right|-\left|\sqrt{a}-1\right|=\left|\underbrace{\sqrt{2{,}5}+1}_{>0}\right|-\left|\underbrace{\sqrt{2{,}5}-1}_{>0}\right|=\sqrt{2{,}5}+1-\sqrt{2{,}5}+1=2.$

Answer: 2.

3.19 Prove that $\left(\frac{11}{5-\sqrt{3}}\right)^2-\left(\frac{5-2\sqrt{5}}{2-\sqrt{5}}\right)^2=\sqrt{\frac{91}{4}+10\sqrt{3}}.$

Evidence:

$$\left(\frac{11}{5-\sqrt{3}}\right)^2-\left(\frac{5-2\sqrt{5}}{2-\sqrt{5}}\right)^2=\left(\frac{11\cdot(5+\sqrt{3})}{(5-\sqrt{3})\cdot(5+\sqrt{3})}\right)^2-\left(\frac{(5-2\sqrt{5})\cdot(2+\sqrt{5})}{(2-\sqrt{5})\cdot(2+\sqrt{5})}\right)^2=$$

$$\text{L.S.}=\left(\frac{11\cdot(5+\sqrt{3})}{25-3}\right)^2-\left(\frac{10+5\sqrt{5}-4\sqrt{5}-10}{4-5}\right)^2=\left(\frac{11\cdot(5+\sqrt{3})}{22}\right)^2-(\sqrt{5})^2=\frac{25+10\sqrt{3}+3}{4}-5=$$

$$=\frac{28+10\sqrt{3}-20}{4}=\frac{8+10\sqrt{3}}{4}=\frac{4+5\sqrt{3}}{2}.$$

$$\text{R.S.}=\sqrt{\frac{91}{4}+10\sqrt{3}}=\sqrt{\frac{91+40\sqrt{3}}{4}}=\frac{\sqrt{91+40\sqrt{3}}}{2}=\frac{\sqrt{16+40\sqrt{3}+75}}{2}=$$

$$=\frac{\sqrt{4^2+2\cdot4\cdot5\sqrt{3}+(5\sqrt{3})^2}}{2}=\frac{\sqrt{(4+5\sqrt{3})^2}}{2}=\frac{\left|4+5\sqrt{3}\right|}{2}=\frac{4+5\sqrt{3}}{2}.$$

Since the left and right sides are equal to the same numeric $\frac{4+5\sqrt{3}}{2}$, then the equality is proved.

In the next two exercises, we will use the idea of raising to some power a selected part of an equality and simultaneously extracting a root from it of the same power..

3.20 Prove equality $\frac{\sqrt{3}-1}{\sqrt{3}+1}=\sqrt[3]{\frac{9-5\sqrt{3}}{9+5\sqrt{3}}}.$

Evidence:

Since the traditional getting rid of irrationality in the denominator of the fraction in this exercise does not lead to the goal, we perform the following transformations on the left side of the equality:

1. Erect it into a cube;
2. At the same time, we extract a cubic root from it.

L.S.

$$\frac{\sqrt{3}-1}{\sqrt{3}+1}=\sqrt[3]{\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right)^3}=\sqrt[3]{\frac{3\sqrt{3}-3\cdot 3\cdot 1+3\cdot\sqrt{3}\cdot 1-1}{3\sqrt{3}+3\cdot 3\cdot 1+3\cdot\sqrt{3}\cdot 1+1}}=\sqrt[3]{\frac{6\sqrt{3}-10}{6\sqrt{3}+10}}=\sqrt[3]{\frac{2\left(3\sqrt{3}-5\right)}{2\left(3\sqrt{3}+5\right)}}=$$

$$=\sqrt[3]{\frac{3\sqrt{3}-5}{3\sqrt{3}+5}}=\sqrt[3]{\frac{\left(3\sqrt{3}-5\right)\cdot\sqrt{3}}{\left(3\sqrt{3}+5\right)\cdot\sqrt{3}}}=\sqrt[3]{\frac{9-5\sqrt{3}}{9+5\sqrt{3}}}=\text{п.ч. } \textit{Рівність доведена}.$$

3.21 Prove equality $\sqrt{8+2\sqrt{10+2\sqrt{5}}}+\sqrt{8-2\sqrt{10+2\sqrt{5}}}=\sqrt{2}\cdot\left(\sqrt{5}+1\right)$.

Evidence:

Since the left side of the expression is positive, to transform it we use the idea of solving the previous exercise:

$$\sqrt{8+2\sqrt{10+2\sqrt{5}}}+\sqrt{8-2\sqrt{10+2\sqrt{5}}}=\sqrt{\left(\sqrt{8+2\sqrt{10+2\sqrt{5}}}+\sqrt{8-2\sqrt{10+2\sqrt{5}}}\right)^2}=$$

$$=\sqrt{\left(\sqrt{8+2\sqrt{10+2\sqrt{5}}}\right)^2+2\sqrt{8+2\sqrt{10+2\sqrt{5}}}\cdot\sqrt{8-2\sqrt{10+2\sqrt{5}}}+\left(\sqrt{8-2\sqrt{10+2\sqrt{5}}}\right)^2}=$$

$$=\sqrt{\sqrt{8+2\sqrt{10+2\sqrt{5}}}+2\sqrt{\left(8+2\sqrt{10+2\sqrt{5}}\right)\cdot\cdot\left(8-2\sqrt{10+2\sqrt{5}}\right)}+8-2\sqrt{10+2\sqrt{5}}}=$$

$$\text{L.S.}=\sqrt{16+2\sqrt{8^2-\left(2\sqrt{10+2\sqrt{5}}\right)^2}}=\sqrt{16+2\sqrt{64-4\left(10+2\sqrt{5}\right)}}=\sqrt{16+2\sqrt{64-40-8\sqrt{5}}}=$$

$$=\sqrt{16+2\cdot\sqrt{24-8\sqrt{5}}}=\sqrt{16+2\sqrt{4\cdot 6-4\cdot 2\sqrt{5}}}=\sqrt{16+4\sqrt{6-2\sqrt{5}}}=\sqrt{16+4\sqrt{\left(\sqrt{5}-1\right)^2}}=$$

$$=\sqrt{16+4\left|\sqrt{5}-1\right|}=\sqrt{16+4\sqrt{5}-4}=\sqrt{12+4\sqrt{5}}=\sqrt{2\left(6+2\sqrt{5}\right)}=\sqrt{2\left(\sqrt{5}+1\right)^2}=\sqrt{2}\cdot\left|\underbrace{\sqrt{5}+1}_{>0}\right|=$$

$$=\sqrt{2}\cdot\left(\sqrt{5}+1\right)=\textit{п.ч.}$$

Self-study assignments:

Find the value of an expression:

3.22 $\sqrt[3]{64\cdot 125}$. Answer: 20.

3.23 $\sqrt[4]{16\cdot 81\cdot 625}$. Answer: 30.

3.24 $\sqrt[6]{4\cdot 16}$. Answer: 2.

3.25 $\sqrt[4]{313^2-312^2}$. Answer: 5.

3.26 $2\cdot\sqrt{40\cdot\sqrt{12}}+3\sqrt{5\cdot\sqrt{48}}-2\sqrt[4]{75}-4\sqrt{15\cdot\sqrt{27}}$. Answer: 0.

3.27 $\left(\sqrt{3-2\cdot\sqrt{2}}+\sqrt{3+2\sqrt{2}}\right)^2$. Answer: 8.

3.28 $\dfrac{1}{4+2\sqrt{3}}+\dfrac{1}{4-2\sqrt{3}}.$ Answer: 2.

3.29 $\dfrac{1}{\sqrt{7}-\sqrt{6}}-\dfrac{3}{\sqrt{6}-\sqrt{3}}-\dfrac{4}{\sqrt{7}+\sqrt{3}}.$ Answer: 0.

3.30 Simplify expression: $\sqrt{3+2\sqrt{2}}-\sqrt{3-2\sqrt{2}}.$ Answer: 2.

3.31 Find the value of an expression: $\sqrt{\dfrac{\sqrt{x}}{\sqrt{x}+5}+\dfrac{25-15\sqrt{x}}{x+10\sqrt{x}+25}+\dfrac{10}{\sqrt{x}+5}}$ at x=26.

Answer: 1.

3.32 Simplify expression: $\sqrt{9+\sqrt{32}};\ \sqrt{6-\sqrt{17-12\sqrt{2}}};$

Prove Equality:

3.33 $\dfrac{\sqrt{3-2\sqrt{2}}}{\sqrt{17-12\sqrt{2}}}-\dfrac{\sqrt{3+2\sqrt{2}}}{\sqrt{17+12\sqrt{2}}}=2;$

3.34 $\dfrac{\sqrt{19}+\sqrt{3}}{2\sqrt{9-\sqrt{65}}}=\dfrac{2\sqrt{9-\sqrt{65}}}{\sqrt{19}-\sqrt{3}}.$

3.35 $\dfrac{2+\sqrt{3}}{\sqrt{2}+\sqrt{2+\sqrt{3}}}+\dfrac{2-\sqrt{3}}{\sqrt{2-\sqrt{2-\sqrt{3}}}}=\sqrt{2}.$

To find:

3.36 $y=\dfrac{2в\sqrt{1+x^2}}{\sqrt{1+x^2}-x}$, if $x=\dfrac{1}{2}\left(\sqrt{\dfrac{a}{в}}-\sqrt{\dfrac{в}{a}}\right)$, *де* $a\rangle 0,\ в\rangle 0.$ Answer: $a+в$.

3.37 $y=\dfrac{\sqrt{(a+x)\cdot(x+в)}+\sqrt{(a-x)\cdot(x-в)}}{\sqrt{(a+x)\cdot(x+в)}-\sqrt{(a-x)\cdot(x-в)}}$, if $x=a\cdot в$, where $a\rangle 0,\ в\rangle 0,\ a\rangle в.$

Answer: $\sqrt{\dfrac{a}{в}}.$

3.38 $y=\dfrac{1}{\sqrt{a+\sqrt{x}}}+\dfrac{1}{\sqrt{a-\sqrt{x}}}$, if $4\cdot(a-1)$, where $a\geq 1$.

Answer: $\dfrac{2}{2-a}$, if $1\leq a\langle 2$; $\dfrac{2\sqrt{a-1}}{a-2}$, if $a\rangle 2.$

3.39 $y=\dfrac{\sqrt{a+x}+\sqrt{x+в}}{\sqrt{a+x}-\sqrt{x-в}}$, if $x=\dfrac{1}{4}z^2+\dfrac{(a-в)^2}{4z^2}-\dfrac{a+в}{2}$, where $z\geq\sqrt{|a-в|}.$

Answer: $\dfrac{|a-в|}{a-в}.$

<u>Определение и свойства логарифма числа</u>

Logarithm of the number N with a base a $(a\rangle 0, a\neq 1)$ is the exponent to which you need to raise the number a to get the number N.

Depending on their base, logarithms can be classified into three large groups:

$$\log_a N = \begin{cases} \lg N, \text{ якщо } a = 10, \text{ – десяткові логарифми}, \\ \ln N, \text{ якщо } a = e, \text{ – натуральні логарифми}, \\ \log_a N, \text{ якщо } a \neq 10 \text{ і } a \neq e, \; e \approx 2{,}7. \end{cases}$$

Useful ratios: $log_a a=1; \quad log_a 1=0.$

Main logarithmic identity: $a^{log_a B} = B.$

Logarithms exist only for positive numbers and have the following properties:

1). $log_a(x_1 \cdot x_2) = \log_a x_1 + \log_a x_2 (x_1 \rangle 0, \; x_2 \rangle 0)$ – *логарифм добутку*;

2). $\log_a \frac{x_1}{x_2} = \log_a x_1 - \log_a x_2 (x_1 \rangle 0, \; x_2 \rangle 0)$ – *логарифм частки*;

3). $\log_a x^k = k \cdot \log_a x (x \rangle 0)$ – *логарифм степеня*;

4). $\log_a \sqrt[n]{x} = \frac{1}{n} \log_a x (x \rangle 0)$ – *логарифм кореня*;

5). $\log_a B = \frac{\log_c B}{\log_c a} \; (c \rangle 0, \; c \neq 1)$ – *формула переходу від однієї основи логарифмів до іншої*;

6). $\log_{a^m}(B^p) = \frac{p}{m} \log_a B.$

Expressing the logarithm in terms of the logarithms of its components is called the logarithm. The inverse of the logarithm is called potentiation.

At first, it is advisable to solve exercises on the application of the basic logarithmic identity, which are good propaedeutics for solving logarithmic equations and inequalities.

For example: $3^{\log_3 4} = 4;$

$$16^{\log_2 7} = (2^4)^{\log_2 7} = (2^{\log_2 7})^4 = 7^4 = 2401;$$

$$2^{\log_8 125} = \left(8^{\frac{1}{3}}\right)^{\log_8 125} = (8^{\log_8 125})^{\frac{1}{3}} = 125^{\frac{1}{3}} = (5^3)^{\frac{1}{3}} = 5;$$

$$4^{\log_8 27} = \left(8^{\frac{2}{3}}\right)^{\log_8 27} = (8^{\log_8 27})^{\frac{2}{3}} = 27^{\frac{2}{3}} = (3^3)^{\frac{2}{3}} = 3^2 = 9.$$

Solution of more complex exercises of this type, for example, to calculate the value of the expression: $36^{\log_6 5} + 10^{1-\lg 2} - 3^{\log_9 36}.$

Solution:

We calculate the value of each of the three members of this expression separately:

$$36^{\log_6 5} = (6^2)^{\log_6 5} = (6^{\log_6 5})^2 = 5^2 = 25;$$

$$10^{1-\lg 2} = 10 \cdot 10^{-\lg 2} = \frac{10}{10^{\lg 2}} = \frac{10}{2} = 5;$$

$$3^{\log_9 36} = \left(9^{\frac{1}{2}}\right)^{\log_9 36} = (9^{\log_9 36})^{\frac{1}{2}} = 36^{\frac{1}{2}} = (6^2)^{\frac{1}{2}} = 6.$$

Means, 25+5 – 6=24. Answer: 24.

3.40 Calculate: $10^{\frac{2}{\log_3 10}} + 3^{\frac{1}{\log_6 3}}.$

Solution:

We use the formula for the transition from one base of logarithms to another:

$\log_3 10 = \frac{\lg 10}{\lg 3} = \frac{1}{\lg 3}$; $\log_6 3 = \frac{\log_3 3}{\log_3 6} = \frac{1}{\log_3 6}$; In this way,

$$10^{\frac{2}{\frac{1}{\lg 3}}} + 3^{\frac{1}{\frac{1}{\log_3 6}}} = 10^{2\lg 3} + 3^{\frac{1}{\frac{1}{\log_3 6}}} = \left(10^{\lg 3}\right)^2 + 3^{\log_3 6} = 3^2 + 6 = 9 + 6 = 15.$$ Answer: 15.

3.41 Calculate: $\log_2\left(2\cdot tg1^\circ\right) + \log_2\left(2^3\cdot tg3^\circ\right) + \log_2\left(2^5\cdot tg5^\circ\right) + \ldots + \log_2\left(2^{89}\cdot tg89^\circ\right)$.

Solution

Replace the amount of the logarithm logarithms product according to the properties of logarithms:

$$\log_2\left(2\cdot tg1^\circ\right) + \log_2\left(2^3\cdot tg3^\circ\right) + \log_2\left(2^{5}\cdot tg5^\circ\right) + \ldots + \log_2\left(2^{89}\cdot tg89^\circ\right) =$$
$$= \log_2\left(2\cdot tg1^\circ\cdot 2^3\cdot tg3^\circ\cdot 2^5\cdot tg5^\circ \ldots \cdot 2^{89}\cdot tg89^\circ\right) = \log_2\left(2\cdot 2^3\cdot 2^5\cdot\ldots\cdot 2^{89}\cdot tg1^\circ\cdot tg3^\circ\cdot tg5^\circ\cdot\ldots\cdot tg89^\circ\right) =$$
$$= \log_2\left(2^{1+3+5+\ldots+89}\cdot\left(tg1^\circ\cdot tg89^\circ\right)\cdot\left(tg3^\circ\cdot tg87^\circ\right)\cdot\ldots\cdot tg45^\circ\right) =$$
$$= \log_2\left(2^{1+3+5+\ldots+89}\cdot\left(\underbrace{tg1^\circ\cdot ctg1^\circ}_{1}\right)\cdot\left(\underbrace{tg3^\circ\cdot ctg3^\circ}_{1}\right)\cdot\ldots\cdot\underbrace{tg45^\circ}_{1}\right) =$$
$$= \left(1+3+5+\ldots+89\right)\underbrace{\log_2 2}_{1} + \underbrace{\log_2 1}_{0} = \frac{2\cdot 1 + 2\cdot(45-1)}{2}\cdot 45 =$$
$$= \frac{2+2\cdot 44}{2}\cdot 45 = \frac{2(1+44)}{2}\cdot 45 = 45\cdot 45 = 2025.$$

From the formula of the nth term of arithmetic progression find the number of its members:

$a_n = a_1 + d(n-1)$; $a_1 = 1$; $a_2 = 3$; $a_n = 89$.
$d = 3-1 = 2$.
$89 = 1 + 2(n-1)$; $2(n-1) = 89-1$; $2(n-1) = 88$;
$n-1 = \frac{88}{2} = 44$; $n = 44+1 = 45$.

Answer: 2025.

To calculate the sum of the first n terms of the arithmetic progression, we use the formula:

$$Sn = \frac{2a_1 + d(n-1)}{n}\cdot n.$$

3.42 Given: $\log_{70} 5 = a$, $\log_{70} 7 = b$. To find $\log_{70} 16$.

Solution:

Let us transform the required logarithm:

$$\log_{70} 16 = \log_{70} 2^4 = 4\cdot\log_{70} 2 = 4\log_{70}\frac{70}{35} = 4\log_{70}\frac{70}{5\cdot 7} = 4\cdot\left(\log_{70} 70 - \log_{70} 5 - \log_{70} 7\right) = 4\cdot(1-a-b).$$

Answer: $4\cdot(1-a-b)$.

3.43 Given: $\log_6 3 = a$, $\log_6 5 = b$. To find $\log_{15} 12$.

Solution:

Expression $\log_{15} 12$ replace the logarithm with the base 6:

$$\log_{15} 12 = \frac{\log_6 12}{\log_6 15} = \frac{\log_6\frac{36}{3}}{\log_6(3\cdot 5)} = \frac{\log_6\frac{6^2}{3}}{\log_6(3\cdot 5)} = \frac{\log_6 6^2 - \log_6 3}{\log_6 3 + \log_6 5} = \frac{2\cdot 1 - \log_6 3}{\log_6 3 + \log_6 5} = \frac{2-a}{a+b}.$$

Answer: 2.

3.44 Given: $\log_{12}5=a,\ \log_{12}18=b.$ To find $\log_{40}54.$

Solution:

Let's move on to logarithms with base 12:

$$\log_{40}54=\frac{\log_{12}54}{\log_{12}40}=\frac{\log_{12}(18\cdot 3)}{\log_{12}(5\cdot 8)}=\frac{\log_{12}18+\log_{12}3}{\log_{12}5+\log_{12}8}=\frac{b+\log_{12}3}{a+\log_{12}2^3}=$$

$$=\frac{b+\log_{12}3}{a+3\log_{12}2}=\frac{b+\log_{12}\sqrt[3]{\frac{18^2}{12}}}{a+3\log_{12}\sqrt[3]{\frac{12^2}{18}}}=\frac{b+\frac{1}{3}\left(\log_{12}18^2-\log_{12}12\right)}{a+3\cdot\frac{1}{3}\left(\log_{12}12^2-\log_{12}18\right)}=$$

$18=2\cdot 9=2\cdot 3^2;\quad 18^2=\left(2\cdot 3^2\right)^2;$

$12=4\cdot 3=2^2\cdot 3;\quad 12^2=\left(2^2\cdot 3\right)^2;$

$\frac{18^2}{12}=\frac{2^2\cdot 3^4}{2^2\cdot 3}=3^3;\quad \frac{12^2}{18}=\frac{2^4\cdot 3^2}{2\cdot 3^2}=2^3;$

$3=\sqrt[3]{\frac{18^2}{12}};\quad 2=\sqrt[3]{\frac{12^2}{18}}.$

$$=\frac{b+\frac{1}{3}\left(2\log_{12}18-\log_{12}12\right)}{a+\left(2\log_{12}12-\log_{12}18\right)}=$$

$$=\frac{b+\frac{1}{3}\cdot(2\cdot b-1)}{a+(2\cdot 1)-b}=$$

$$=\frac{b+\frac{2b-1}{3}}{a+2-b}=$$

$$=\frac{5b-1}{3a-3b+6}.$$

Answer: $\frac{5b-1}{3a-3b+6}.$

3.45 Given: $\log_{20}50=a,\quad \log_{3}20=b.$ To find $\log_{150}200.$

Solution:

Let's move on to logarithms with base 20:

$$\log_3 20=\frac{\log_{20}20}{\log_{20}3}=\frac{1}{\log_{20}3};\quad \frac{1}{\log_{20}3}=b;\quad \log_{20}3=\frac{1}{b}.$$

$$\log_{150}200=\frac{\log_{20}200}{\log_{20}150}=\frac{\log_{20}\left(50\cdot 2^2\right)}{\log_{20}(50\cdot 3)}=\frac{\log_{20}50+2\log_{20}20}{\log_{20}50+\log_{20}3}=$$

$20=2^2\cdot 5;\ 50=5^2\cdot 2;\ 20^2=2^4\cdot 5^2;$

$\frac{20^2}{50}=\frac{2^4\cdot 5^2}{5^2\cdot 2}=2^3;\ 2=\sqrt[3]{\frac{20^2}{50}}.$

$$=\frac{a+2\log_{20}\sqrt[3]{\frac{20^2}{50}}}{a+\frac{1}{b}}=$$

$$=\frac{a+\frac{2}{3}\cdot\left(2\log_{20}20-\log_{20}50\right)}{a+\frac{1}{b}}=$$

$$=\frac{a+\frac{2}{3}\cdot\left(2\cdot 1-\log_{20}50\right)}{a+\frac{1}{b}}=$$

$$= \frac{a + \frac{4}{3} - \frac{2}{3}a}{\frac{ab+1}{b}} = \frac{\frac{1}{3}a + \frac{4}{3}}{\frac{ab+1}{b}} = \frac{(a+4)\cdot b}{3ab+3} = \frac{ba+4b}{3ab+3}.$$ Answer: $\frac{ab+4b}{3ab+3}$.

Self-study assignments:

3.46 It is known that $lg2 = 0,301$. Calculate $\log_3 2 \cdot \log_4 3 \cdot \log_5 4 \cdot ... \cdot \log_9 8 \cdot \lg 9$.
Answer: 0,301.

Calculate:

3.47 $20 log_4 3 \cdot \log_{81} 64$. Answer: 15.

3.48 $18 \cdot \frac{\log_4 12}{3 + \log_4 27}$. Answer: 6

3.49 $(14 \cdot \log_8 7)^4 \cdot \log_{49} 512$. Answer: 11.

3.50 $57 \cdot \frac{\log_5 200}{6 + \log_5 512}$. Answer: 19.

3.51 $-\log_2\left(\log_2 \sqrt{\sqrt{2}}\right)$. Answer: 2.

3.52 $25^{\log_5 3}$. Answer: 9.

3.53 $3 \cdot \log_{\sqrt{8}} 2 + 2^{-2\cdot\log_{\frac{1}{2}} 2}$. Answer: 6.

3.54 $3 \cdot \log_{\sqrt{64}} 8 + 4^{-2\cdot\log_{\frac{1}{4}} 3}$. Answer: 11.

3.55 $(\log_4 11 + \log_4 23) : \log_8 253$. Answer: 1,5.

3.56 $\log_4 128$. Answer: 3,5.

3.57 $\log_4 в$, if $в = sin\frac{\pi}{6}$. Answer: $-\frac{1}{2}$.

3.58 $3^{\log_3 8} - 2\log_3 2 + \log_3\left(\frac{9}{2}\right)$. Answer: 9.

3.59 $\log_3 \log_2 \left(\sqrt[3]{2^k}\right)^{\frac{1}{2}}$, если $\log_3 K = 10$. Answer: 8.

3.60 $(\log_3 8)\cdot(\log_{17} 3)\cdot(\log_4 17)$. Answer: 1,5

3.61 $8 + \frac{1}{2}\log_{\frac{2}{3}}\left(6 - \sqrt{20}\right) + \frac{1}{2}\log_{\frac{2}{3}}\left(6 + \sqrt{20}\right) - \log_{\frac{2}{3}} 9$. Answer: 10.

3.62 $\log_{a\cdot b} x$, если $\log_a x = 2$, $\log_b x = 3$. Answer: 1,2.

3.63 $\log_{\frac{5}{a}} 25$, если $\log_a 5 = 2$. Answer: 4.

3.64 $(\log_3 2 + \log_2 81 + 4)\cdot(\log_3 2 - 2\log_{18} 2)\cdot \log_2 3 - \log_3 2$. Answer: 2.

To find *x*, if:

3.65 $\log_{\sqrt[3]{32}} x = -1,5$. Answer: $\frac{1}{4\sqrt{2}}$.

3.66 $\log_x \sqrt[4]{\frac{27}{6561}} = -0,75$. Answer: $3 \cdot \sqrt[3]{9}$.

3.67 $x = \log_{\sqrt[3]{2}} 8 \cdot \sqrt[4]{8}$. Answer: 11,25.

3.68 Given: $\log_{14} 7 = a$, $\log_{14} 5 = b$. To find $\log_{35} 28$. Answer: $\frac{2-a}{a+b}$.

3.69 Given: $\log_{10} 16 = a$. To find $\log_{50} 25$. Answer: $\frac{8-2a}{8-a}$.

3.70 Given: $\log_7 12 = a$, $\log_{12} 24 = b$. To find $\log_{168} 54$. Answer: $\frac{8a-5ab}{ab+1}$.

3.71 Given: $\lg 225 = a$, $\lg 60 = b$. To find $\lg 5$. Answer: $\frac{a-2b+4}{4}$.

3.72 Given: $\lg 225 = a$, $\lg 60 = b$. To find $\lg 2$. Answer: $\frac{2b-a}{4}$.

3.73 To find $\log_{54} 168$, if $\log_7 12 = a$, $\log_{12} 24 = b$. Answer: $-\frac{1}{ab}$.

3.74 To find $\log_5 6{,}125$, if $\log_{25} 7 = a$, $\log_2 5 = b$. Answer: $\frac{4ab-3}{b}$.

3.75 To find $\log_{30} 18$, if $\log_3 2 = a$, $\log_5 3 = b$. Answer: $\frac{ab+2b}{1+b+ab}$.

3.76 To find $\lg 45$, if $\log_2 3 = a$, $\log_5 2 = b$. Answer: $\frac{1+2ab}{b+1}$.

3.77 To find $\log_8 9{,}8$, if $\lg 2 = a$, $\lg 7 = b$. Answer: $\frac{a+2b-1}{3a}$.