Section 13
Combinatorial equations and their systems

Finite sets for which the order of elements in them is essential are called ordered.. For example, point *A(1; 3)* and point *B(3; 1)* – ordered sets because A(1; 3)$\neq$B(3; 1).
Any ordered set consisting of *n* elements is called a permutation with *n* elements. Permutations with *n* elements differ from each other only in the order of the elements.
Permutation formula $P_n = n!$ $n! = 1\cdot 2\cdot 3\cdot \ldots \cdot n.$ ($n!$-read: *"n factorial")*
$0! = 1.$
Any ordered subset with *n* elements of a given set *M* containing elements, where $n \le m,$ is called an arrangement from *m* elements to *n*.
Placement is different either by elements or by order of elements.
Denoted $A_m^n.$
Formula: $A_m^n = \underbrace{m\cdot(m-1)\cdot(m-2)\cdot\ldots\cdot(m-n+1)}_{n-\text{множників}}.$
For example: $A_4^3 = 4\cdot 3\cdot 2 = 24.$
Helpful formula $A_m^{k+1} = A_m^k\cdot(m-k).$
Any subset with *n* elements of a given set *M* containing *m* elements is called a combination with *m* elements in *n*. Denoted $C_m^n.$
Formula: $C_m^n = \dfrac{m!}{n!(m-n)!}.$
In the process of solving various combinatorial exercises, such formulas can serve as a good lever:
1). $C_m^n = \dfrac{A_m^n}{P_n}.$ 2). $C_m^n = C_m^{m-n}.$ 3). $C_m^{n+1} = \dfrac{m-n}{n+1}\cdot C_m^n.$
4). $C_m^n + C_m^{n+1} = C_{m+1}^{n+1}.$ 5). $C_m^0 + C_m^1 + C_m^2 + \ldots + C_m^n = 2^m.$ 6). $C_0^0 = 1,\ C_m^0 = 1,\ C_m^m = 1.$
Combinatorics uses only natural numbers.
Combinatorial we assume equations and their systems containing at least one of the sets:
permutations, placement, combinations.
Solve the equation:
1). $C_x^{13} = C_x^7.$

Solution:

By formula (2), we have: $C_x^7 = C_x^{x-7}.$ Then $13 = x-7,\ x = 13+7,\ x = 20.$
Answer: 20.
2). $A_x^3 + C_x^{x-2} = 14\cdot x.$

Solution:

By formula (1), we have: $A_x^3 = C_x^3\cdot P_3.$ Then $C_x^3\cdot P_3 + C_x^{x-x+2} = 14x,$

$$3!\cdot\frac{x!}{3!(x-3)!}+C_x^2=14x,\ \frac{x!}{(x-3)!}+\frac{x!}{2!(x-2)!}=14x,$$

$$\frac{(x-3)!(x-2)(x-1)\cdot x}{(x-3)!}+\frac{(x-2)!(x-1)\cdot x}{1\cdot 2\cdot (x-2)!}=14x\Big|:x,$$

$$(x-2)\cdot(x-1)+\frac{x-1}{2}=14\Big|\cdot 2,\ 2\cdot\left(x^2-x-2x+2\right)+x-1=28,\ 2x^2-6x+4+x-1-28=0,$$

$$2x^2-5x-25=0,\ D=25+200=225.\ x_1=\frac{5-15}{4}=-\frac{10}{4}=-2{,}5\notin N;\ x_2=\frac{5+15}{4}=5.$$

Answer: 5.

3). $12C_{x+3}^{x-1}=55\cdot A_{x+1}^2.$

Solution:

$$12C_{x+3}^{x+3-x+1}=55\cdot A_{x+1}^2;\ 12C_{x+3}^4=55\cdot A_{x+1}^2;$$

$$12\cdot\frac{(x+1)!}{4!(x+1-4)!}=55\cdot(x+1)\cdot x;\ \frac{(x+1)!}{2\cdot(x-3)!}=55\cdot(x+1)\cdot x;$$

$$\frac{(x-3)!(x-2)(x-1)x\cdot(x+1)}{2(x-3)!}=55(x+1)\cdot x\Big|\cdot 2;$$

$$(x-2)(x-1)x(x+1)=110(x+1)x\Big|:(x+1)\cdot x\neq 0;$$

$$(x-2)\cdot(x-1)=110,\ x^2-x-2x+2-110=0,\ x^2-3x-108=0,\ D=9+432=441=21^2;$$

$$x_1=\frac{3-21}{2}=-9\notin N;\ x_2=\frac{3+21}{2}=12.$$

Answer: 12. *(maybe Answer 8).*

4). $\dfrac{1}{C_4^x}-\dfrac{1}{C_5^x}=\dfrac{1}{C_6^x}.$

Solution:

It's obvious that $0<x\le 4$.

On the left side of the equation, we reduce to a common denominator:

$$\frac{C_5^x-C_4^x}{C_4^x\cdot C_5^x}=\frac{1}{C_6^x};\ \frac{\dfrac{5!}{x!(5-x)!}-\dfrac{4!}{x!(4-x)!}}{\dfrac{4!}{x!(4-x)!}\cdot\dfrac{5!}{x!(5-x)!}}=\frac{1}{C_6^x};$$

$$\frac{\dfrac{5!(4-x)!-4!(5-x)!}{x!(5-x)!(4-x)!}}{\dfrac{4!5!}{x!\cdot x!(4-x)!(5-x)!}}=\frac{1}{C_6^x};\ \frac{x!(5!(4-x)!-4!(5-x)!)}{4!5!}=\frac{1}{C_6^x};$$

$$\frac{x!4!(4-x)!(5-5+x)}{4!5!}=\frac{1}{C_6^x};\ \frac{x!(4-x)!x}{5!}=\frac{1}{\dfrac{6!}{x!(6-x)!}};$$

$$\frac{x!(4-x)!x}{5!}=\frac{x!(6-x)!x}{6!}\Big|:(x!\cdot(4-x)!);$$

$$\frac{x}{5!}=\frac{(5-x)(6-x)}{6!}\Big|\cdot 6!;\ 6x=(5-x)(6-x);\ 6x=30-5x-6x+x^2;\ x^2-17x+30=0;$$

By Vieta's theorem, we have: $x_1 = 2;\quad x_2 = 15 -$ does not satisfy the condition $0 < x \le 4.$

Answer: 2.

$$\begin{cases} C_x^y = C_x^{y+2}, \\ C_x^2 = 153. \end{cases}$$

Solution:

Let's solve the second equation of the system:

$$C_x^2 = \frac{x!}{2!(x-2)!} = \frac{1\cdot 2\cdot \ldots \cdot (x-2)\cdot (x-1)\cdot x}{2!1\cdot 2\cdot \ldots \cdot (x-2)} = \frac{(x-1)\cdot x}{2};\quad \frac{(x-1)\cdot x}{2} = 153\big|\cdot 2;\quad (x-1)\cdot x = 306;$$

$$x^2 - x - 306 = 0.$$

By Vieta's theorem $x_1 = 18,\ x_2 = -17,\ -17 \notin N.$

Substitute $x = 18$ into the first equation of the system:

$$C_{18}^y = C_{18}^{y+2};\quad \frac{18!}{y!(18-y)!} = \frac{18!}{(y+2)!(18-y-2)!}\big|:18!\quad \frac{1}{y!(18-y)!} = \frac{1}{(y+2)!(16-y)!};$$

From the equality of fractions with the same numerators, the equality of the denominators follows, that is:

$$y!(18-y)! = (y+2)!(16-y)!\big|: y!$$

$$(18-y)! = (y+1)\cdot (y+2)\cdot (16-y)!\big|:(16-y)!$$

$$(18-y)\cdot (17-y) = (y+1)\cdot (y+2);\quad 306 - 18y - 17y + y^2 = y^2 + 2y + y + 2;$$

$$-38y = -304;\quad y = \frac{-304}{-38};\quad y = 8.$$

Answer: (18; 8).

$$\begin{cases} A_{5x}^{y-3} : A_{5x}^{y-2} = 1:7, \\ C_{5x}^{y-2} : C_{5x}^{y-3} = 7:4. \end{cases}$$

Solution:

Let us give this system of equations the following form:

$$A_{5x}^{y-3} = C_{5x}^{y-3}\cdot P_{y-3} = C_{5x}^{y-3}\cdot 1\cdot 2\cdot \ldots \cdot (y-5)\cdot (y-4)\cdot (y-3);$$

$$A_{5x}^{y-2} = C_{5x}^{y-2}\cdot P_{y-2} = C_{5x}^{y-2}\cdot 1\cdot 2\cdot \ldots \cdot (y-3)\cdot (y-2).$$

$$\begin{cases} \dfrac{C_{5x}^{y-3}\cdot 1\cdot 2\cdot \ldots \cdot (y-3)}{C_{5x}^{y-2}\cdot 1\cdot 2\cdot \ldots \cdot (y-3)\cdot (y-2)} = \dfrac{1}{7}, \\ \dfrac{C_{5x}^{y-2}}{C_{5x}^{y-3}} = \dfrac{7}{4}. \end{cases}\quad \begin{cases} \dfrac{C_{5x}^{y-3}\cdot}{C_{5x}^{y-2}\cdot (y-2)} = \dfrac{1}{7}, \\ \dfrac{C_{5x}^{y-2}}{C_{5x}^{y-3}} = \dfrac{7}{4}. \end{cases} \to C_{5x}^{y-2} = \frac{7}{4}\cdot C_{5x}^{y-3}.$$

$$\begin{cases} C_{5x}^{y-3} = \dfrac{1}{7}, \\ \dfrac{7}{4}C_{5x}^{y-3}\cdot (y-2). \end{cases}\qquad \frac{1}{\frac{7}{4}(y-2)} = \frac{1}{7};\quad \frac{7}{4}(y-2) = 7;\quad y-2 = 4,\quad y = 6.$$

We substitute in the second equation:

$$\frac{C_{5x}^{6-2}}{C_{5x}^{6-3}} = \frac{7}{4};\quad \frac{C_{5x}^4}{C_{5x}^3} = \frac{7}{4};\quad 4\cdot C_{5x}^4 = 7\cdot C_{5x}^3.$$

$$4\cdot\frac{(5x)!}{4!(5x-4)!}=7\cdot\frac{(5x)!}{3!(5x-3)!}\Big|:(5x)!\quad \frac{4}{1\cdot2\cdot3\cdot4\cdot(5x-4)!}=\frac{7}{3!(5x-3)!}\Big|\times 3!$$

$$\frac{1}{(5x-4)!}=\frac{7}{(5x-3)!};\quad (5x-3)!=7(5x-4)!\,|:(5x-4)!\quad 5x-3=7;\quad 5x=10;\quad x=2.$$

Answer: (2; 6).

Consider the equation that reduces to the system:

$C_{n+1}^{m+1}:C_{n+1}^{m}=5:5.$

Solution:

$$\begin{cases}C_{n+1}^{m+1}:C_{n+1}^{m}=5:5,\\ C_{n+1}^{m}:C_{n+1}^{m-1}=5:3.\end{cases}\quad \begin{cases}\dfrac{C_{n+1}^{m+1}}{C_{n+1}^{m}}=\dfrac{5}{5},\\ \dfrac{C_{n+1}^{m}}{C_{n+1}^{m-1}}=\dfrac{5}{3}.\end{cases}\quad \begin{cases}\dfrac{\dfrac{(n+1)!}{(m+1)!(n+1-m-1)!}}{\dfrac{(n+1)!}{m!(n+1-m)!}}=1,\\ \dfrac{\dfrac{(n+1)!}{(m+1)!(n+1-m)!}}{\dfrac{(n+1)!}{(m-1)!(n+1-m+1)!}}=\dfrac{5}{3}.\end{cases}\quad \begin{cases}\dfrac{m!(n-m+1)!}{(m+1)!(n-m)!}=1,\\ \dfrac{(m-1)!(n-m+2)!}{m!(n-m+1)!}=\dfrac{5}{3}.\end{cases}$$

$$\begin{cases}\dfrac{n-m+1}{m+1}=1,\\ \dfrac{n-m+2}{m}=\dfrac{5}{3}.\end{cases}\quad \begin{cases}n-m+1=m+1,\\ 3n-3m+6=5n.\end{cases}\quad \begin{cases}n=2m,\\ 8m=3n+6.\end{cases}\quad 8m=3\cdot2m+6,\ \ 2m=6,\ \ m=3.\ \ n=2\cdot3=6.$$

Answer: (3; 6).

Self-study assignments:

$x^2\cdot C_{x-1}^{x-4}=A_4^2\cdot C_{x+1}^3-x\cdot C_{x-1}^{x-4}.$ Answer: 6.

$C_{x+8}^{x+3}=5\cdot A_{x+6}^3.$ Answer: 17.

$\dfrac{A_{x+2}^{n+2}\cdot P_{x-n}}{P_x}=110.$ Answer: $x=9$ at $n=\{1;2;3;4;5;6;7;8\}$.

$\dfrac{P_{x+2}}{A_x^n\cdot P_{x-n}}=132.$ Answer: $x=10$ at $n=9!$

$\dfrac{C_{2x}^{x+1}}{C_{2x+1}^{x-1}}=0{,}6.$ Answer: 7.

$C_{20}^{x}=C_{20}^{3x-12}.$ Answer: 6; 8.

$\dfrac{1}{C_4^x}-\dfrac{1}{C_5^x}=\dfrac{1}{C_6^x}.$ Answer: 2.

$\begin{cases}A_x^y:A_x^{y-1}=10,\\ C_x^y:C_x^{y-1}=\dfrac{5}{3}.\end{cases}$ Answer: (15; 6).

$\begin{cases}C_x^y=C_x^{y+2},\\ C_x^2=133.\end{cases}$ Answer: (6; 3).

$$\begin{cases} C_x^{y+1} = 2{,}5x, \\ C_{x-1}^{y} = 10. \end{cases}$$

Answer: (4; 3).

$$C_{x+1}^{y-1} : C_{x+1}^{y} : C_{x+1}^{y+1} = 3 : 5 : 5.$$

Answer: (6; 3).