Section 6
Irrational equations

Irrational are equations in which the variable is under the root sign.

In the school curriculum, they are solved only in the set of real numbers..

To solve an irrational equation, it is necessary by means of identical transformations to reduce it to a rational. This is achieved by raising both sides of the equation to a power, or by introducing a new variable. Since when raising both sides of the equation to an even power, the appearance of extraneous roots is possible, then to "filter" them, it is advisable to substitute all the found values of the roots in the original equation.

Consider some methods for solving irrational equations, namely:

1). The introduction of a new variable;

2). Applying abbreviated multiplication formulas;

3). Introduction of conjugate expressions;

4). Subtraction of the common factor in parentheses.

Solve the equation: $\sqrt{5x-1}=x+1.$

Solution:

Square both sides of the equation: $5x-1=x^2+2x+1;\ \ x^2-3x+2=0;$

By the theorem of Vieta: $x_1=1;\ \ x_2=2.$

Check: if $x=1,$ then $\sqrt{5\cdot 1-1}=1+1;\ \ \ 2=2.$

$1-$ root of the equation. If $x=2,$ то $\sqrt{5\cdot 2-1}=2+1;\ \ \ 3=3.$

$2-$ root of the equation.

Answer: 1; 2.

Solve the equation: $\sqrt{5x-3}=\sqrt{5-3x}.$

Solution:

Raise both sides of the square: $5x-3=5-3x;\ \ \ 5x+3x=5+3;\ \ \ 8x=8;\ \ \ x=1.$

Check: $\sqrt{5\cdot 1-3}=\sqrt{5-3\cdot 1};\ \ \ \sqrt{2}=\sqrt{2}-$ true equality.

$1-$ root of the equation.

Answer: 1.

Solve the equation: $\sqrt{6x-2}-\sqrt{4x-3}-\sqrt{x-2}=0.$

Solution:

$\sqrt{6x-2}-\sqrt{4x-3}-\sqrt{x-2}=0.$ Let's raise both parts to a square:

$6x-2-2\cdot\sqrt{6x-2}\cdot\sqrt{4x-3}+4x-3=x-2;\ \ \ 10x-5-2\sqrt{(6x-2)\cdot(4x-3)}=x-2;$

$10x-5-x+2=2\sqrt{(6x-2)(4x-3)};\ \ \ 10x-3=2\cdot\sqrt{(6x-2)(4x-3)};$

Squaring both sides of the equation again: $81x^2-54x+9=4\cdot(6x-2)\cdot(4x-3);$

$81x^2-54x+9=4(24x^2-18x-8x+6);\ \ 81x^2-54x+9-96x^2+72x+32x-24=0;$

$-15x^2+50x-15=0|:(-5);\ \ \ 3x^2-10x+3=0;\ \ \ D=100-36=64=8^2;$

$x_1=\frac{10-8}{6}=\frac{1}{3};\ x_2=\frac{10+8}{6}=3;$ Check: if $x=\frac{1}{3},$ то $\sqrt{6\cdot\frac{1}{3}-2}-\sqrt{4\cdot\frac{1}{3}-3}-\sqrt{\frac{1}{3}-2}\neq 0.$

Well then $\frac{1}{3}$ extraneous root.

If $x=3,$ then $\sqrt{6\cdot 3-2}-\sqrt{4\cdot 3-3}-\sqrt{3-2}=0;$ $3-$ root of the equation.
Answer: 3.
Solve the equation: $\sqrt[3]{3-x}+\sqrt[3]{3+x}=1.$

Solution:

Let's write the formula for the sum of cubes in this form:
$(a+b)^3=a^3+3a^2b+3ab^2+b^3=a^3+b^3+3ab\cdot(a+b);$
Let us raise both sides of the equation to a cube and apply the above identity:
$3-x+3+x+3\cdot\sqrt[3]{3-x}\cdot\sqrt[3]{3+x}\cdot\left(\sqrt[3]{3-x}+\sqrt[3]{3+x}\right)=1;$ $6+3\cdot\sqrt[3]{(3-x)\cdot(3+x)}\cdot 1=1;$
$3\cdot\sqrt[3]{9-x^2}=-5;$
We raise both sides of the equation in a cube: $27\cdot\left(9-x^2\right)=-125;$ $9-x^2=\frac{-125}{27};$
$x^2=9+\frac{125}{27};$ $x^2=\frac{243+125}{27};$ $x^2=\frac{368}{27}=\frac{16\cdot 23}{9\cdot 3};$ $x_2=\frac{4}{9}\sqrt{69}.$
Answer: $-\frac{4}{9}\cdot\sqrt{69};$ $\frac{4}{9}\sqrt{69}.$
Solve the equation $5\sqrt[6]{x-6}=\sqrt[3]{x-6}+6.$

Solution:

Let's introduce a new variable $\sqrt[6]{x-6}=t,$ $\left(\sqrt[6]{x-6}\right)^2=t^2,$ $\sqrt[3]{x-6}=t^2.$
Then the original equation will have the form $t^2-5t+6=0.$
Its roots by Vieta's theorem $t_1=2;$ $t_2=3.$ Then $\sqrt[6]{x-6}=2,$ $\left(\sqrt[6]{x-6}\right)^6=2^6;$ $x-6=64;$
$x=64+6;$ $x=70;$
$\sqrt[6]{x-6}=3;$ $\left(\sqrt[6]{x-6}\right)^6=3^6;$ $x-6=729;$ $x=729+6;$ $x=735.$
Check: If $x=70,$ then $5\cdot\sqrt[6]{70-6}=5\cdot\sqrt[6]{64}=5\cdot 2=10;$ $\sqrt[3]{70-6}+6=\sqrt[3]{64}+6=4+6=10;$
$10=10.$
$70-$ root of the equation. If $x=735,$ then $5\cdot\sqrt[6]{735-6}=5\cdot\sqrt[6]{729}=5\cdot 3=15;$
$\sqrt[3]{735-6}+6=\sqrt[3]{729}+6=9+6=15.$ $15=15.$ $735-$ root of the equation.
Answer: 70; 735.
Solve the equation : $\sqrt{3x^2-2x+15}+\sqrt{3x^2-2x+8}=7.$

Solution:

We denote $\sqrt{3x^2-2x+15}-\sqrt{3x^2-2x+8}=A.$
We multiply the original equation and the just formed:
$\left(\sqrt{3x^2-2x+15}+\sqrt{3x^2-2x+8}\right)\cdot\left(\sqrt{3x^2-2x+15}-\sqrt{3x^2-2x+8}\right)=7A;$
$3x^2-2x+15-\left(3x^2-2x+8\right)=7A;$ $3x^2-2x+15-3x^2+2x-8=7A;$ $7A=7;$ $A=1.$
Let us solve the following system of equations $\begin{cases}\sqrt{3x^2-2x+15}+\sqrt{3x^2-2x+8}=7,\\ \sqrt{3x^2-2x+15}-\sqrt{3x^2-2x+8}=1.\end{cases}$
Adding these equations, we get $2\sqrt{3x^2-2x+15}=8\,|:2;$ $\sqrt{3x^2-2x+15}=4.$
Let's square both sides of the equation $3x^2-2x+15=16;$
$3x^2-2x-1=0;$ $D=4+12=16;$ $x_1=\frac{2-4}{6}=-\frac{2}{6}=-\frac{1}{3};$ $x_2=\frac{2+4}{6}=1.$
Verification shows that both found numbers satisfy the equation.

Answer: $-\frac{1}{3}$; 1.

There are many irrational equations, the Solution of which is greatly simplified when the radical expression is factorized with the subsequent removal of the common factor outside the brackets. For example, solve the equation:

$\sqrt{2x^2+7x-15}+\sqrt{2x^2-3x}=\sqrt{4x^2+4x-15}.$

Solution:

$\sqrt{2x^2+7x-15}+\sqrt{2x^2-3x}-\sqrt{4x^2+4x-15}=0.$ Expand each of radicands $2x^2+7x-15,\quad 2x^2-3x,\quad 4x^2+4x-15.$ For the first and third polynomials, we apply the factorization formula for a square trinomial: $ax^2+bx+c=a\cdot(x-x_1)\cdot(x-x_2)$, where x_1 and x_2 – the roots of this trinomial.

$D=7^2+4\cdot2\cdot15=49+120=169,\quad x_1=\frac{-7-\sqrt{169}}{2\cdot2}=\frac{-7-13}{4}=-5;\quad x_2=\frac{-7+13}{4}=\frac{3}{2}$, then

$2x^2+7x-15=2\cdot\left(x-\frac{3}{2}\right)\cdot(x+5)=(2x-3)\cdot(x+5).$ $D=16+16\cdot15=256,$

$x_1=\frac{-4-16}{8}=-\frac{20}{8}=-\frac{5}{2};\quad x_2=\frac{-4+16}{8}=\frac{12}{8}=\frac{3}{2}$, then

$4x^2+4x-15=4\cdot\left(x-\frac{3}{2}\right)\cdot\left(x+\frac{5}{2}\right)=2\cdot\left(x-\frac{3}{2}\right)\cdot2\left(x+\frac{5}{2}\right)=(2x-3)\cdot(2x+5).$

$2x^2-3x=x\cdot(2x-3).$ The equation takes the form:

$\sqrt{(2x-3)\cdot(x+5)}+\sqrt{x\cdot(2x-3)}-\sqrt{(2x-3)\cdot(2x+5)}=0.$

$\sqrt{2x-3}\cdot\sqrt{x+5}+\sqrt{x}\cdot\sqrt{2x-3}-\sqrt{2x-3}\cdot\sqrt{2x+5}=0;\quad \sqrt{2x-3}\cdot\left(\sqrt{x+5}+\sqrt{x}-\sqrt{2x+5}\right)=0;$

$\sqrt{2x-3}=0,$ or $\left(\sqrt{x+5}+\sqrt{x}-\sqrt{2x+5}\right)=0;$

$2x-3=0;\qquad \sqrt{x+5}+\sqrt{x}=\sqrt{2x+5};$

$x=1{,}5;$ Let's square both sides of the equation: $x+5+2\sqrt{(x+5)\cdot x}+x=2x+5;$

$2\cdot\sqrt{(x+5)\cdot x}=0;\quad \left[\begin{array}{l}x+5=0,\\x=0\end{array}\right.\quad \left[\begin{array}{l}x=-5,\\x=0.\end{array}\right.$

Verification shows that for $x=0$ the first and third radicals have no meaning, and the numbers $1{,}5$ and -5 satisfy the equation.

Answer: -5; $1{,}5$.

The solution of equations with mutually inverse quantities is reduced after introducing a new variable to the quadratic equation.

Solve the equation: $\sqrt[3]{\frac{5-x}{x+3}}+\sqrt[3]{\frac{x+3}{5-x}}=2.$

Solution:

R.O.V.V.: $x\neq(-3);\ x\neq5.$ We denote $\sqrt[3]{\frac{5-x}{x+3}}=t,$ then $\sqrt[3]{\frac{x+3}{5-x}}=\frac{1}{t};\quad t+\frac{1}{t}=2;$

$\frac{t^2-2t+1}{t}=0;\quad \begin{cases}t^2-2t+1=0,\\t\neq0.\end{cases}\quad \begin{cases}(t-1)^2=0,\\t\neq0.\end{cases}\quad \begin{cases}t=1,\\t\neq0.\end{cases}\quad \sqrt[3]{\frac{5-x}{x+3}}=1;\quad \frac{5-x}{x+3}=1;$

$5-x=x+3;\quad 5-x=x+3;\quad -2x=-2;\quad x=1.$

Verification shows that 1 satisfies the original equation.

Answer: 1.
Sometimes it is useful to replace an irrational equation with a system of equations.
Solve the equation: $2\cdot\sqrt{3x-2}=18-2\cdot\sqrt{x+7}$.

Solution:

$2\cdot\sqrt{3x-2}+2\cdot\sqrt{x+7}=18\,|:2;\quad \sqrt{3x-2}+\sqrt{x+7}=9\quad (A).$

Having solved the system of inequalities, we find R.O.V.V. of equation.

$\begin{cases}3x-2\geq 0,\\ x+7\geq 0.\end{cases}\quad \begin{cases}3x\geq 2,\\ x\geq -7.\end{cases}\quad \begin{cases}x\geq \frac{2}{3};\\ x\geq 7.\end{cases}\quad \left[\frac{2}{3};+\infty\right)-$ domain of equation.

We introduce new variables, namely: $\sqrt{3x-2}=m,\ m\geq 0.\quad \sqrt{x+7}=n,\ n\geq 0.$
Raising both sides in the square, we get: $3x-2=m^2;\quad x+7=n^2|\times(-3);$

$$+\begin{cases}-3x-21=-3n^2\\ 3x-2=m^2\end{cases}$$

$$-23=m^2-3n^2$$

$\begin{cases}m+n=9;\\ m^2-3n^2=-23.\end{cases}\quad \begin{cases}n=9-m;\\ m^2-3(9-m)^2+23=0.\end{cases}\quad \begin{cases}n=9-m;\\ m^2-3\cdot(81-18m+m^2)+23=0.\end{cases}$

$m_1=5;\ m_2=22;\ n_1=9-5=4;\ n_2=9-22=-13.$

Back to the substitution: $\sqrt{3x-2}=5;\ 3x-2=25;\ 3x=27;\ x=9.$ $n=-13-$ extraneous root because it does not satisfy the condition $n\geq 0$. A direct check shows that the number 9 − root of the equation.
Answer:9.
Often there are such irrational equations, for the solution of which it is sufficient to replace only one of several radicals with a new variable. Solve the equation:
$\sqrt[3]{x-2}+\sqrt{x+6}=6.$

Solution:

R.O.V.V.: $x+6\geq 0,\ x\geq -6.$ Let's introduce a new variable $\sqrt[3]{x-2}=y,$ then
$x-2=y^3,\quad x=y^3+2,\quad \sqrt{x+6}=\sqrt{y^3+2+6}=\sqrt{y^3+8},\quad y+\sqrt{y^3+8}=6,\quad \sqrt{y^3+8}=6-y.$
After squaring both sides of the equation, we get: $y^3+8=36-12y+y^2;$
$y^3-y^2+12y-28=0.$
Free term divisors of equation:
$28M\{\pm 1,\ \pm 2,\ \pm 4,\ \pm 7,\ \pm 14;\ \pm 28\}$. If $y=2,$ то $2^3-2^2+12\cdot 2-28=8-4+24-28=0.$
$y=2-$ root of the equation.

$$\begin{array}{r|l} y^3-y^2+12y-28 & y-2 \\ \cline{2-2} -\underline{y^3-2y^2} & y^2+y+14 \\ y^2+12y & \\ -\underline{y^2-2y} & \\ 14y-28 & \\ -\underline{14y-28} & \\ 0. & \end{array}$$

The equation $y^2 + y + 14 = 0$ has no real roots, because its discriminant is negative. $\sqrt[3]{x-2} = 2,\ x-2=8,\ x=10.$ Direct verification shows that 10 is the root of the equation.

Let us show another, in our opinion, a unique way to solve irrational equations of the above type: $\sqrt[3]{x+24} = \sqrt{12-x}$.

Solution:

We introduce the following notation:

$\sqrt[3]{x+24} = y_1;\ \ \sqrt{12-x} = y_2$ тоді $y_1 = y_2$. Raising the first equality to a cube, the second to a square and taking into account the third equality, we obtain the system of equations:

$$\begin{cases} {y_1}^3 = x+24, \\ {y_2}^2 = 12-x, \\ y_1 = y_2. \end{cases}$$

$$+\begin{array}{l} {y_2}^3 = x+24, \\ {y_2}^2 = 12-x \\ \hline \end{array}$$

${y_2}^3 + {y_2}^2 = 36,\ \ {y_2}^2 + {y_2}^2 - 36 = 0.$

$\{\pm 1,\ \pm 2,\ \pm 3,\ \pm 4,\ \pm 6,\ \pm 9,\ \pm 12,\ \pm 18,\ \pm 36\}-$ free term divisors of equation.

Substituting them sequentially into the equation, we make sure that the number 3 is its root: $3^3 + 3^2 - 36 = 27 + 9 - 36 = 0.$

${y_2}^3 + {y_2}^2 - 36$

$$\begin{array}{l|l} {y_2}^3 + {y_2}^2 - 36 & y_2 - 3 \\ \cline{2-2} -\underline{{y_2}^3 - 3{y_2}^2} & {y_2}^2 + 4y_2 + 12 \\ \quad -4{y_2}^2 - 36 & \\ \quad\ \ \underline{4{y_2}^2 - 12y_2} & \\ \qquad\quad -12y_2 - 36 & \\ \qquad\qquad \underline{12y_2 - 36} & \\ \qquad\qquad\quad 0. & \end{array}$$

${y_2}^2 + 4y_2 + 12 = 0,\ \ D = 16 - 48 < 0,\ \ y_2 \in \varnothing.$

In the second equality instead of y_2 substitute the number 3:

$\sqrt{12-x} = 3,\ \ 12 - x = 9,\ \ x = 3.$

Number 3 satisfies the original equation.

Answer: 3.

In some equations, it is useful to transfer only one term from one part to another, but such that the sums of the coefficients of the variables in both sides of the equation are the same.

Solve the equation: $\sqrt{8x+1} + \sqrt{3x-5} = \sqrt{7x+4} + \sqrt{2x-2}.$

Solution:

$\sqrt{8x+1} - \sqrt{2x-2} = \sqrt{7x+4} - \sqrt{3x-5};$ 8+2=10; 7+3=10.

After squaring both sides of the equation, we have:

$8x+1-2\sqrt{(8x+1)\cdot(2x-2)} + 2x - 2 = 7x + 4 - 2\cdot\sqrt{(7x+4)\cdot(3x-5)} + 3x - 5;$

$10x-1-2\sqrt{(8x+1)\cdot(2x-2)}=10x-1-2\cdot\sqrt{(7x+4)\cdot(3x-5)};$
$2\sqrt{(8x+1)\cdot(2x-2)}=-2\cdot\sqrt{(7x+4)\cdot(3x-5)};$
$\sqrt{(8x+1)\cdot(2x-2)}=\sqrt{(7x+4)\cdot(3x-5)};\ (8x+1)\cdot(2x-2)=(7x+4)\cdot(3x-5);$
$16x^2-16x+2x-2=21x^2-35x+12x-20;\ 5x^2+9x+18=0;\ \ 5x^2-9x-18=0;$
$D=81+360=441;$
$x_1=\dfrac{9-21}{10}=-1{,}2;\ \ x_2=\dfrac{9+21}{10}=3.$
By direct verification, we make sure that only the number 3 is the root of the original equation.
Answer: 3.
I would like to acquaint those working with the "Workshop on solving problems in elementary mathematics" with an original way of solving irrational equations of this kind $\sqrt{x+3-4\cdot\sqrt{x-1}}+\sqrt{x+8-6\cdot\sqrt{x-1}}=1.$

Solution:

Let's introduce a new variable $\sqrt{x-1}=y$, $y\geq 0$. $\left(\sqrt{x-1}\right)^2=y^2$; $x-1=y^2$; $x=y^2+1$.
Then this equation will have the form:
$\sqrt{y^2+1+3-4y}+\sqrt{y^2+1+8-6y}=1;$
$\sqrt{y^2-4y+4}+\sqrt{y2-6y+9}=1;$
$\sqrt{(y-2)^2}+\sqrt{(y-3)^2}=1;$
$|y-2|\ \ +|y-3|=1.$
We equate the submodular differences to zero, solve the resulting equations and, taking into account the inequality $y\geq 0$ we denote the found roots of the equation on the number line:
y−2=0; y=2.
y−3=0; y=3.
On $[0;2)$ $y-2<0$, and therefore $|y-2|=-(y-2)=2-y;\ \ y-3<0,$
and therefore $|y-3|=-(y-3)=3-y;$
Equation (1) takes the form: $2-y+3-y=1;\ \ -2y=-4;\ \ y=2;\ \ 2\notin[0;2).$
The equation has no roots on this interval. On the interval $[2;3)$
$y-2>0,\ |y-2|=y-2;\ \ y-3<0,\ \ |y-3|=-(y-3)=3-y.\ \ y-2+3-y=1,\ \ 1=1$ for any value of the variable y c $[2;3)$. $2\leq\sqrt{x-1}<3;\ \ 4\leq x-1<9;\ \ 5\leq x<10;\ \ x\in[5;10).$
On $[3;+\infty)$; $y-2>0;\ |y-2|=y-2.\ \ y-3>0;\ |y-3|=y-3.\ \ y-2+y-3=1;\ \ 2y=6;$
$y=3;\ \ 3\in[3;+\infty)$. 3 – root of the equation (1). $\sqrt{x-1}=3;\ \ x-1=9;\ \ x=10.$
Thus, the solutions to this equation are (line segment) $[5;10]$.
Answer: $[5;10]$.
The method of solving equations of this type is of great interest:
$\dfrac{10x^3-3x^2-2x+1}{\sqrt{x-2}}=0.$

Solution:

The condition of equality of the fraction to zero implies the following system: $\begin{cases} 10x^3 - 3x^2 - 2x + 1 = 0; \\ \sqrt{x-2} \neq 0. \end{cases}$

It is easy to see that $x \neq 0,$ and therefore we divide both sides of the equation by x^3 followed by the introduction of a new variable. $10x^3 - 3x^2 - 2x + 1 = 0 \,\big|: x^3;$

$10 - \frac{3}{x} - \frac{2}{x^2} + \frac{1}{x^3} = 0$. Introducing a new variable $\frac{1}{x} = y;$ $10 - 3y - 2y^2 + y^3 = 0;$

$y^3 - 2y^2 - 3y + 10 = 0$ (1). $10 M \{\pm 1;\ \pm 2;\ \pm 5;\ \pm 10\}$. If $y = -2,$ then

$(-2)^3 - 2\cdot(-2)^2 - 3\cdot(-2) + 10 = -8 - 8 + 6 + 10 = 0;$ $0=0$; It means that $y = -2 -$ root of the equation (1). $\frac{1}{x} = -2;$ $x = -\frac{1}{2}.$

$$
\begin{array}{r|l}
y^3 + 2y^2 - 3y + 10 & y + 2 \\ \cline{2-2}
-\ \underline{y^3 + 2y^2} \qquad\qquad & y^2 + 4y + 5 \\
-4y^2 - 3y \qquad & \\
-\ \underline{-4y^2 - 8y} \qquad & \\
5y + 10 & \\
-\ \underline{5y + 10} & \\
0. &
\end{array}
$$

$y^2 - 4y + 5 = 0$; $D = 16 - 20 = -4 < 0;$ $y \in \varnothing$ in area of R.

Check: if $x = -\frac{1}{2},$ then $10\cdot\left(-\frac{1}{2}\right)^3 - 3\cdot\left(-\frac{1}{2}\right)^2 - 2\cdot\left(-\frac{1}{2}\right) + 1 = -\frac{10}{8} - \frac{3}{4} + 1 + 1 = 0.$

Answer: $-\frac{1}{2}.$

Prove that the equation $\sqrt{x^4 + x - 2} + \sqrt[4]{x^4 + x - 2} = 6$ has a single positive root and find it.

Solution:

Let's introduce a new variable: $\sqrt[4]{x^4 + x - 2} = t;$ $\sqrt{x^4 + x - 2} = t^2.$

The original equation has the form: $t^2 + t - 6 = 0.$ By Vieta's theorem: $t_1 = -3;$ $t_2 = 2.$

$\sqrt[4]{x^4 + x - 2} = -3,$ $x \in \varnothing.$ $\sqrt[4]{x^4 + x - 2} = 2;$ $x^4 + x - 2 = 16;$ $x^4 - 16 + x - 2 = 0;$

$(x^2 - 4)\cdot(x^2 + 4) + (x - 2) = 0;$

$(x - 2)\cdot(x + 2)\cdot(x^2 + 4) + (x - 2) = 0;$ $(x - 2)\cdot((x + 2)\cdot(x^2 + 4) + 1) = 0.$

$\left[\begin{array}{l} x - 2 = 0, \\ x^3 + 4x + 2x^2 + 8 + 1 = 0. \end{array}\right.$ $\left[\begin{array}{l} x = 2, \\ x^3 + 2x^2 + 4x + 9 = 0 \end{array}\right.$ це рівняння додатних коренів не має.

Find negative divisors of the number 9: $\{-1;\ -3;\ -9\}.$

If $x = -1,$ then $(-1)^3 + 2\cdot(-1)^2 + 4\cdot(-1) + 9 = -1 + 2 - 4 + 9 \neq 0.$

If $x = -3,$ then $(-27) + 18 - 12 + 9 \neq 0.$

If $x = -9,$ then $729 + 162 - 36 + 9 \neq 0.$

Check: if $x = 2,$ то $\sqrt{2^4 + 2 - 2} + \sqrt[4]{2^4 + 2 - 2} = \sqrt{2^4} + \sqrt[4]{2^4} = 4 + 2 = 6;$ $6 = 6.$

Answer: $x=2-$ single positive root.

Self-study assignments:

Solve the equation:

6.1 $\sqrt{20-x}=7-\sqrt{x+5}.$ Answer: 4; 11.

6.2 $\sqrt{x-3-4+\sqrt{2x+1}}=0$ Answer: 4.

6.3 $\sqrt{x^2+5x-2}-\sqrt{x^2-3x+3}=3.$ Answer:2.

6.4 $\sqrt[3]{\dfrac{12-2x}{x-1}}+\sqrt[3]{\dfrac{x-1}{12-2x}}=\dfrac{5}{2}.$ Answer: 2; $\dfrac{97}{17}$.

6.5 $\sqrt{x^2+x-1}=x.$ Answer: 1.

6.6 $\sqrt{x^2+x-1}=\sqrt{x}.$ Answer: 1.

6.7 $\sqrt[3]{x^3-19}=x-1.$ Answer: $-2;\ 3.$

6.8 $\sqrt{x+6}-\sqrt{x+1}=\sqrt{2x-5}.$ Answer: 3.

6.9 $2x^2+6-2\sqrt{2x^2-3x+2}=3x+12.$ Answer: $-2;\ 3{,}5.$

6.10 $\sqrt{\dfrac{x-5}{x+2}}+\sqrt{\dfrac{x-4}{x+3}}=\dfrac{7}{x+2}\cdot\sqrt{\dfrac{x+2}{x+3}}.$ Answer: 6.

6.11 $\dfrac{1}{\sqrt{x-3}}-\sqrt{x-3}=\sqrt{x-6}.$ Answer: $\varnothing$.

6.12 $\sqrt[8]{x^5}+2\sqrt{x}=3\cdot\sqrt[4]{x^3}.$ Answer: 0; 1.

6.13 $\sqrt{x^2-3x+11}-4x^2+12x=11.$ Answer: 1; 2.

6.14 $\dfrac{(x-1)\cdot\sqrt[3]{10-x}-(10-x)\cdot\sqrt[3]{x-1}}{\sqrt[3]{x-1}-\sqrt[3]{10-x}}.$ Answer: 2; 9.

6.15 $\dfrac{1}{x}+\dfrac{1}{\sqrt{13-x^2}}=\dfrac{5}{6}.$ Answer: $\dfrac{-\sqrt{481}+13}{10};\ \ 2;\ 3.$

6.16 $\sqrt[3]{2x+13}-\sqrt[3]{2x-13}=2.$ Answer: $-7;\ 7.$

6.17 $\sqrt[4]{x+62}+\sqrt[4]{6-x}=3\sqrt{2}.$ Answer: $-58;\ 2.$

6.18 $\sqrt[4]{x+14}+\sqrt[4]{3-x}=3.$ Answer: $-13;\ 2.$

6.19 $x\cdot\sqrt{x^2+15}-\sqrt{x}\cdot\sqrt[4]{x^2+15}=2.$ Answer: 1.

6.20 $x^2-4x+6=\sqrt{2x^2-8x+12}.$ Answer: 2.

6.21 $3x^2+15x+2\sqrt{x^2+5x+1}=2.$ Answer: $0;\ -5.$

6.22 $\sqrt{2x+3}+\sqrt{x+1}=3x+2\cdot\sqrt{2x^2+5x+3}=16.$ Answer: 3.

6.23 $\sqrt[4]{x+8}-\sqrt[4]{x-8}=2.$ Answer: 8.

6.24 $\dfrac{1}{\sqrt{x}+\sqrt[3]{x}}+\dfrac{1}{\sqrt{x}-\sqrt[3]{x}}=\dfrac{1}{3}.$ Answer: 64.

6.25 $\sqrt{2x^2+8x+6}+\sqrt{x^2-1}=2x+2.$ Answer: $-1;\ 1.$

6.26 $\sqrt{x-2}+\sqrt{4-x}=x^2-6x+11.$ Answer: 3.

6.27 $\sqrt[3]{(2-x)^2}+\sqrt[3]{(7+x)^2}-\sqrt[3]{(2-x)\cdot(7+x)}=3.$ Answer: $-6;\ 1.$

6.28 $x^2 + x \cdot \sqrt{x+1} - 2 \cdot (x+1) = 0.$ Answer: $2 + 2\sqrt{2}$; $\dfrac{1+\sqrt{5}}{2}$.

6.29 $\dfrac{(x-a)\cdot\sqrt{x-a} + (x-b)\cdot\sqrt{x-b}}{\sqrt{x-a} + \sqrt{x-b}} = a - b$, if $a > b$. Answer: $x_1 = a$; $x_2 = \dfrac{4a-b}{3}$.

6.30 $\dfrac{\sqrt{x} + \sqrt[3]{x}}{\sqrt{x} - \sqrt[3]{x}} = 3.$ Answer: 64.

6.31 $\sqrt[3]{x+1} = \sqrt{x-3}.$ Answer: 7.

6.32 $\sqrt[3]{x-1} + \sqrt[3]{27-14x} = 1.$ Answer: 1.

6.33 $\sqrt{2x^2 + 8x + 6} + \sqrt{x^2 - 1} = 2x + 2.$ Answer: $x \in (-2;0) \cup (1;+\infty)$.

6.34 $\sqrt{x - 2\cdot\sqrt{x-1}} + \sqrt{x + 3 - 4\cdot\sqrt{x-1}} = 1.$ Answer: $x \in (-\infty;-1) \cup (0;2)$.

6.35 $\sqrt[4]{x+8} - \sqrt[4]{x-8} = 2.$ Answer: 8.