Section 11
Equations with parameters

Many difficulties arise when solving equations with parameters.

This is due to many reasons, among them:

а) insufficiently broad and complete definition of the concept of a parameter;

б) lack of clear definition of the parameter;

в) very weak technique for solving equations with parameters.

The concept of a parameter arose in technology, where it characterizes a certain essential property of a parameter (for example, lamp parameters), area, phenomena, etc. Then it "migrated" to other sciences, including mathematics, where it is treated as another variable in the equation that differs from the one whose value needs to be found.

In the equation $\frac{3x+2}{a-1}=8,$ x – аргумент, a – параметр,

that is, a quantity that characterizes, sets the permissible values.

Solving an equation with parameters means finding all solutions of this equation for certain permissible values of parameters.

$$\frac{3}{ax+2}-\frac{2}{3x+a}=0.$$

Solution:

We multiply both sides of the equation by the expression: $(3x+a)(ax+2)$:

$$\frac{3\cdot(3x+a)(ax+2)}{ax+2}-\frac{2\cdot(3x+a)(ax+2)}{3x+a}=0;$$

$$3(3x+a)-2(ax+2)=0.$$

We express argument x via parameter a:

$$9x+3a-2ax-4=0;\ \ 9x-2ax=4-3a;\ \ x(9-2a)=4-3a;$$

$$x=\frac{4-3a}{9-2a}=\frac{-(3a-4)}{-(2a-9)}=\frac{3a-4}{2a-9}.$$

It is easy to establish at what values of the parameter *a* the last and naturally given equation has no roots - equate the denominator $2a-9$ to zero, that is $2a-9=0,$ $2a=9,$ $a=4{,}5.$

Hence, for $a=4{,}5$ $x\in\varnothing.$

At the next stage of the solution, we establish at what values of the parameter *a* the equation has roots. This happens when $a\neq 4{,}5,$ then $2a\neq 9$ and the equation has one root $x=\frac{3a-4}{2a-9}.$ Since the right-hand side of this equality contains the multivalued parameter *a*, then it is necessary to determine those values of this parameter at which $x=\frac{3a-4}{2a-9}$ is the root of this equation. These will be those values *a*, which do not turn the denominators of the original equation into zero, that is, for which $ax+2\neq 0$ and $3x+a\neq 0.$

We substitute into these irregularities $x=\dfrac{3a-4}{2a-9}$:

$$ax+2=a\cdot\frac{3a-4}{2a-9}+2=\frac{3a-4a+4a-18}{2a-9}=\frac{3a^2-18}{2a-9}=\frac{3\left(a^2-6\right)}{2a-9}\neq 0;\ \frac{3\left(a^2-6\right)}{2a-9}\neq 0,$$

then when

$$a^2-6\neq 0\quad a^2\neq 6;\ \left[\begin{array}{l} a\neq -\sqrt{6}; \\ a\neq \sqrt{6}. \end{array}\right.$$

Similarly $3x+a=3\cdot\dfrac{3a-4}{2a-9}+a=\dfrac{9a-12+2a^2-9a}{2a-9}=\dfrac{2a^2-12}{2a-9}=\dfrac{2\left(a^2-6\right)}{2a-9}\neq 0$ at $a\neq\pm\sqrt{6}$.

Answer: at $a=4{,}5; a=-\sqrt{6}; a=\sqrt{6}\quad x\in\varnothing$; at $a\neq 4{,}5; a\neq -\sqrt{6}; a\neq\sqrt{6}\quad x=\dfrac{3a-4}{2a-9}$.

$$\frac{1}{x-a}+\frac{1}{x-b}=\frac{1}{a}+\frac{1}{b}.$$

Solution:

In this equation x - argument, a and b - parameters.

Provided that $a\neq 0$ and $b\neq 0$ multiply both sides of the equation by the expression: $(x-a)\cdot(x-b)\cdot a\cdot b$:

$$\frac{1\cdot(x-a)\cdot(x-b)\cdot a\cdot b}{x-a}+\frac{1\cdot(x-a)\cdot(x-b)\cdot a\cdot b}{x-b}=\frac{1\cdot(x-a)\cdot(x-b)\cdot a\cdot b}{a}+\frac{1\cdot(x-a)\cdot(x-b)\cdot a\cdot b}{b};$$

$$(x-b)\cdot a\cdot b-(x-a)\cdot a\cdot b=(x-a)\cdot(x-b)\cdot b+(x-a)\cdot(x-b)\cdot a;$$

$$xab-ab^2+xab-a^2b=x^2b-xb^2-axb+ab^2+ax^2-abx-a^2b+a^2b;$$

Let us group the terms of the equation so that it is possible to take out of the brackets from one group x^2, on the other - x and the third group would be without an argument, that is, form a quadratic equation:

$$\left(-x^2b-ax^2\right)+\left(xab+xab+xab+xab+xb^2+a^2x\right)-ab^2-a^2b-ab^2-a^2b=0;$$

$$-(a+b)\cdot x^2+\left(a^2+b^2+4ab\right)\cdot x-2ab(a+b)=0\,|\cdot(-1);$$

$$(a+b)\cdot x^2-\left(a^2+b^2+4ab\right)x+2ab\cdot(a+b)=0.\ (A)$$

For a quadratic equation to exist, it is necessary that its leading coefficient is nonzero, that is $a+b\neq 0$:

1). Let the $b=-a$. Then the original equation takes the form $\dfrac{1}{x-a}+\dfrac{1}{x+a}=\dfrac{1}{a}+\dfrac{1}{-a}$;

$$\frac{1}{x-a}+\frac{1}{x+a}=\frac{1}{a}-\frac{1}{a};\ \frac{1}{x-a}+\frac{1}{x+a}=0;$$

$\dfrac{x+a+x-a}{(x-a)\cdot(x+a)}=0;\ 2x=0;\ x=0-$ root of the equation.

2). Let the $a+b\neq 0$, i.e $b\neq -a$, then the equation *(A)* – square equation:

$$D=\left(a^2+b^2+4ab\right)^2-4(a+b)\cdot ab\cdot(a+b)=$$

$$=a^4+b^4+16a^2b^2+2a^2b^2+8a^3b+8ab^3-8ab\left(a^2+2ab+b^2\right)=$$

$$=a^4+b^4+16a^2b^2+2a^2b^2+8a^3b+8ab^3-8a^3b-16a^2b^2-8ab^3=$$

$$=a^4+b^4+2a^2b^2=\left(a^2+b^2\right)^2.$$

$$x_1 = \frac{a^2+b^2+4ab-\sqrt{\left(a^2+b^2\right)^2}}{2(a+b)} = \frac{a^2+b^2+4ab-\left|a^2+b^2\right|}{2(a+b)} = \frac{a^2+b^2+4ab-a^2-b^2}{2(a+b)} = \frac{2a-b}{a+b};$$

$$x_2 = \frac{a^2+b^2+4ab+a^2+b^2}{2(a+b)} = \frac{2a^2+2b^2+4ab}{2(a+b)} = \frac{2(a+b)^2}{2(a+b)} = a+b;$$

Answer: if $a \neq b, b \neq 0, |b| \neq |a|,$ then $x_1 = \frac{2ab}{a+b};\ x_2 = a+b;$

If $b=-a,$ then $x=0;$ If $a=0$ or $b=0,$ then $x \in \varnothing.$

Let us show the application of equations with parameters to solving, for example, geometric problems:

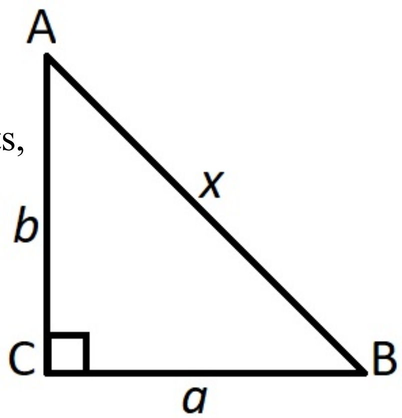
The sum of the hypotenuse and one of the legs is m, and the sum of the hypotenuse and the other leg is n.

Find the hypotenuse.

Solution:

Let the x – the hypotenuse of the triangle, then $a \neq x = m, b+x=n.$

From here $a = m-x, b = n-x.$



By the Pythagorean theorem, we have:

$x^2 = a^2+b^2;\ x^2 = (m-x)^2+(n-x)^2;$

Because the $a>0, b>0, x>0$ as the lengths of the segments, then $m>n, n>0,\ 0<x<m,\ 0<x<n.$

We solve the resulting equation:

$x^2 = m^2-2mx+x^2+n^2-2nx+x^2;$

$x^2-2\cdot(m+n)\cdot x+\left(m^2+n^2\right)=0;$

$D = 4\cdot(m+n)^2-4\cdot\left(m^2+n^2\right)=$

$=4m^2+8mn+4n^2-4m^2-4n^2=8mn.$

As $8mn>0,$ then the equation has two roots:

$$x_1 = \frac{2(m+n)-\sqrt{8mn}}{2} = \frac{2(m+n)-2\sqrt{2mn}}{2} = m+n-\sqrt{2mn};$$

$x_2 = m+n+\sqrt{2mn}-$ does not satisfy the condition of the problem, because $m+n+\sqrt{2mn}>m.$

Let us find out under what condition the root $x_2>0.$

$x_2<m,\ x_2<n,$ i.e

$m+n-\sqrt{2mn}<m,$	$m+n-\sqrt{2mn}<n,$
$n-\sqrt{2mn}<0,$	$m-\sqrt{2mn}<0,$
$n<\sqrt{2mn},$	$m<\sqrt{2mn},$
$n^2<2mn\,\vert:n,$	$m^2<2mn\,\vert:m$
$n<2m\,\vert:2,$	$m<2n\ (B);$
$\frac{n}{2}<m\ \ (C);$	

With inequalities *(B)* and *(C)* the inequality follows $\frac{n}{2}<m<2n\,\Big|:n,\ \frac{1}{2}<\frac{m}{n}<2.$

Answer: at $\frac{1}{2} < \frac{m}{n} < 2$ the task has a unique solution $x = m + n - \sqrt{2mn}$.

$\sqrt{x+a} + \sqrt{x+b} = \sqrt{2x+a-b}$.

Solution:

We square both sides of the equation, we get:

$x + a + 2\sqrt{(x+a)\cdot(x-b)} + x - b = 2x + a - b$;

$2\sqrt{(x+a)\cdot(x+b)} = 0,\ \sqrt{(x+a)\cdot(x+b)} = 0$.

After squaring, we have:

$(x+a)\cdot(x+b) = 0\ \left[\begin{array}{l} x+a=0, \\ x-b=0 \end{array}\right. \left[\begin{array}{l} x_1=-a, \\ x_2=b. \end{array}\right.$

It is necessary to determine at which *a* and *b* x_1 and x_2 will be the roots of this equation.

Substitute $-a$ and b into the equation:

$\sqrt{-a+a} + \sqrt{-a-b} = \sqrt{-2a+a-b};\ 0 + \sqrt{-a-b} = \sqrt{-a-b}$.

The left and right sides of the equation make sense when $-a-b \geq 0,\ -a \geq b \,|: (-1)$,

$a \leq -b$.

Therefore, for $a \leq -b$ $x = -a -$ root of the equation.

$\sqrt{b+a} + \sqrt{b-b} = \sqrt{2b+a-b}, \sqrt{b+a} = \sqrt{b+a},\ b + a \geq 0,\ a \geq -b,\ x = b -$ root of the equation.

Answer: at $a \leq -b,\ x = -a;$ at $a \geq -b,\ x = b.$ $\quad a^x + a^{-x} = 2c.$

Solution:

$a > 0, a \neq 1.\ a \in (0;1) \cup (1;+\infty),\quad c \in R.$

These are exponential equations, $x -$ argument, a and $c -$ parameters.

$a^x + a^{-x} = 2c \,|\cdot a^2,\ (a^x)^2 + 1 = 2c\cdot a^2,\ (a^x)^2 - 2c\cdot a^x + 1 = 0.$

Formed a quadratic equation a^x. We denote $a^x = t$.

$t^2 - 2ct + 1 = 0;\ D = (-2c)^2 - 4\cdot 1 = 4\cdot(c^2-1) \geq 0.$

The equation has roots:

$$t_1 = \frac{2c - \sqrt{4(c^2-1)}}{2} = c - \sqrt{c^2-1} \text{ i } t_2 = c + \sqrt{c^2-1}.$$

Taking into account the substitution, we have a set of equations:

$$\left[\begin{array}{l} a^x = c - \sqrt{c^2-1}, \\ a^x = c + \sqrt{c^2-1}. \end{array}\right.$$

Let us find out at what values of the parameter *C* the set as well as the original equation have solutions.

At $c < 1$ totality has no solutions.

At $c > 1$ totality has two solutions:

$$\log_a a^x = \log_a\left(c - \sqrt{c^2-1}\right),\ x_1 = \log_a\left(c - \sqrt{c^2-1}\right);$$

$$\log_a a^x = \log_a\left(c + \sqrt{c^2-1}\right),\ x_1 = \log_a\left(c + \sqrt{c^2-1}\right);$$

At $c = 1$

$$a^x = 1 - \sqrt{1^2-1};\ a^x = 1;\ a^x = a^0;\ x = 0;$$

$$a^x = 1+\sqrt{1^2-1};\; a^x = 1;\; x = 0;$$

Answer: at $c<1,\; x\in\varnothing;$ at $c>0,\; x_1 = \log_a\left(c-\sqrt{c^2-1}\right);\; x_2 = \log_a\left(c+\sqrt{c^2-1}\right);$ at $c=1,\; x=0,$

$$\frac{\log_{a^2\sqrt{x}} a}{\log_{2x} a} + \log_{ax} a\cdot\log_{\frac{1}{a}} 2x = 0.$$

Solution:

$a\in(0;1)\cup(1;+\infty).$ Reduce all the logarithms of the equation to the base *a:*

$$\log_{a^2\sqrt{x}} a = \frac{\log_a a}{\log_a a^2\sqrt{x}} = \frac{1}{\log_a a^2\sqrt{3}};\; \log_{2x} a = \frac{\log_a a}{\log_a 2x} = \frac{1}{\log_a 2x};$$

$$\log_{ax} a = \frac{\log_a a}{\log_a ax} = \frac{1}{\log_a ax};\; \log_{\frac{1}{a}} 2x = \frac{\log_a 2x}{\log_a \frac{1}{a}} = \frac{\log_a 2x}{\log_a a^{-1}} = -\log_a 2x.$$

Then the original equation will have the form:

$$\frac{1}{\log_a a^2\sqrt{3}} : \frac{1}{\log_a ax} + \frac{1}{\log_a ax}\cdot\left(-\log_a 2x\right) = 0;$$

$$\frac{\log_a 2x}{\log_a a^2\sqrt{3}} - \frac{\log_a 2x}{\log_a ax} = 0;\; \frac{\log_a 2x}{\log_a a^2 + \log_a\sqrt{3}} - \frac{\log_a 2x}{\log_a a + \log_a x} = 0;$$

$$\frac{\log_a 2x}{2+\frac{1}{2}\log_a 3} - \frac{\log_a 2x}{1+\log_a x} = 0\Big|\cdot\frac{1}{\log_a 2x};\; \frac{1}{2+\frac{1}{2}\log_a 3} - \frac{1}{1+\log_a x} = 0;$$

$\dfrac{1}{2+\frac{1}{2}\log_a 3} = \dfrac{1}{1+\log_a x};$ Hence it follows that $1+\log_a x = 2+\dfrac{1}{2}\log_a 3;$

$$\log_a x = 1+\frac{1}{2}\log_a 3;\; \log_a x = \log_a a + \log_a\sqrt{3};\; \log_a x = \log_a a\sqrt{3},\; x = a\sqrt{3}.$$

Answer: at $a=(0;1)\cup(1;\infty),\; x = a\sqrt{3}.$

$$\sin(x-\alpha) - \sin x = \sin\alpha;$$

Solution:

$$\sin(x-\alpha) - \sin x - \sin\alpha = 0;\; \sin\alpha\cdot\frac{x-\alpha}{2} - (\sin x + \sin\alpha) = 0;$$

$$2\sin\frac{x-\alpha}{2}\cdot\cos\frac{x-\alpha}{2} - 2\sin\frac{x+\alpha}{2}\cdot\cos\frac{x-\alpha}{2} = 0;$$

$$2\cos\frac{x-\alpha}{2}\cdot\left(\sin\frac{x-\alpha}{2} - \sin\frac{x+\alpha}{2}\right) = 0;$$

$$2\cos\frac{x-\alpha}{2}\cdot 2\cos\frac{\frac{x-\alpha}{2}+\frac{x+\alpha}{2}}{2}\cdot\sin\frac{\frac{x-\alpha}{2}+\frac{x+\alpha}{2}}{2} = 0;$$

$$4\cdot\cos\frac{x-\alpha}{2}\cdot\cos\frac{x}{2}\cdot\sin\left(-\frac{\alpha}{2}\right) = 0;\; -4\sin\frac{\alpha}{2}\cdot\cos\frac{\alpha}{2}\cdot\cos\frac{x-\alpha}{2} = 0;\; -4\neq 0;$$

$$\left[\begin{array}{l}\sin\frac{\alpha}{2}=0,\\ \cos\frac{\alpha}{2}=0,\\ \cos\frac{x-\alpha}{2}=0.\end{array}\right.\quad \left[\begin{array}{l}\frac{\alpha}{2}=k\pi,\\ \frac{\alpha}{2}=\frac{\pi}{2}+\pi n,\\ \frac{\alpha}{2}=\frac{\pi}{2}+\pi n.\end{array}\right.\quad \left[\begin{array}{l}\alpha=2k\pi,\\ \alpha=\pi+2k\pi,\\ x=\pi+2k\pi+\alpha\end{array}\right.\quad x=\alpha+\pi(2k+1).$$

Answer: at $\alpha=0$ $x=(2k+1)\cdot\pi$, at $\alpha\neq 0$ $x=\alpha+(2k+1)\cdot\pi$, $k\in Z$.

$$a\sin x-b\cos\frac{x}{2}=0.$$

Solution:

$$a2\sin\frac{x}{2}\cos\frac{x}{2}-b\cos\frac{x}{2}=0;\ \cos\frac{x}{2}\left(2a\sin\frac{x}{2}-b\right)=0;$$

$$\left[\begin{array}{l}\cos\frac{x}{2}=0,\\ 2a\sin\frac{x}{2}-b=0\end{array}\right.\quad \left[\begin{array}{l}\frac{x}{2}=\frac{\pi}{2}+\pi n,\\ \frac{x}{2}=(-1)^n\arcsin\frac{b}{2a}\end{array}\right.\quad \left[\begin{array}{l}x=\pi+2\pi n,\\ x=2\cdot(-1)^n\arcsin\frac{b}{2a}+2\pi n.\end{array}\right.$$

If $\left|\frac{b}{2a}\right|\le 1$, i.e $|b|\le 2|a|$, then $x=(-1)^n\cdot 2\arcsin\frac{b}{2a}+2\pi n.$

If $\left|\frac{b}{2a}\right|>0$, then $x\in\varnothing$.

Answer: if $a\neq 0$ and $|b|\le 2a$, then $x=(2n+1)\pi$ and $x=(-1)^n 2\arcsin\frac{b}{2a}+2\pi n.$

University applicants are often offered equations with parameters that are difficult to solve using the above method. Here the following way can come to the rescue: consider the parameter as an argument, and consider the variable as a coefficient.

For example: $\frac{1}{2n+nx}-\frac{1}{2x-x^2}=\frac{2(n+3)}{x^3-4x}.$

Solution:

Parameter n is considered a variable, and the variable $x-$ coefficient.
We rewrite this equation as follows:

$$\frac{1}{n(2+x)}-\frac{2(n+3)}{x(x^2-4)}=\frac{1}{2x-x^2};\ \frac{1}{n(x+2)}-\frac{2(n+3)}{x(x-2)(x+2)}=\frac{1}{2x-x^2};$$

Range of valid values: $x\neq 0,\ x\neq 2,\ x\neq -2.$

$$\frac{x(x-2)-2n(n+3)}{nx(x-2)(x+2)}=\frac{1}{2x-x^2};$$

Let's use the main property of proportion:

$$x(x-2)(2x-x^2)-2n(n+3)(2x-x^2)=nx(x-2)(x+2);$$
$$x(x-2)(2x-x^2)=nx(x-2)(x+2)+2n(n+3)(2x-x^2);$$
$$x(x-2)x(2-x)=nx(x^2-4)+2nx(n+3)(2-x);$$
$$-x^2(x-2)^2=nx(x^2-4)-2nx(n+3)(x-2);$$
$$-x^2(x-2)^2=nx^3-4nx+(-2n^2x-6nx)(x-2);$$
$$-x^2(x-2)^2=nx^3-4nx-2n^2x^2+4n^2x+6nx^2-12nx;$$
$$-x^2(x-2)^2=nx^3+6nx^2-2n^2x^2-16nx+4n^2x;$$

$$-x^2(x-2)^2 = nx^3+6nx^2-16nx+4n^2x-2n^2x^2;$$
$$-x^2(x-2)^2 = \left(4n^2x-2n^2x^2\right)+nx^3+6nx^2-16nx;$$
$$-x^2(x-2)^2 = 2\left(2x-x^2\right)n^2+\left(x^3+6x^2-16x\right)n;$$
$$2\left(2x-x^2\right)n^2+\left(x^3+6x^2-16x\right)n+x^2(x-2)^2 = 0.$$

A quadratic equation was formed with respect to n.

$$D = \left(x^3+6x^2-16x\right)^2-4\cdot 2\cdot\left(2x-x^2\right)\cdot x^2(x-2)^2 =$$
$$= x^6+36x^4+256x^2+12x^5-32x^4-192x^3-\left(16x^3+8x^4\right)\left(x^2-4x+4\right)=$$
$$= x^6+4x^4+12x^5-192x^3+256x^2-16x^5+64x^4-64x^3-8x^6+32x^5-32x^4 =$$
$$= -7x^6+28x^5+36x^4-256x^3+256x^2 = x^2\left(-7x^4-64x^3+36x^2-256x+256\right).$$

$$\frac{1}{n(x+2)}-\frac{1}{x(2-x)}-\frac{2(n+3)}{x(x-2)(x+2)} = 0;$$
$$\frac{x(x-2)+n(x+2)-2n(n+3)}{nx(x+2)(x-2)} = 0;$$
$$\begin{cases} x^2-2x+nx+2n-2n^2-6n = 0, \\ nx(x+2)(x-2)\neq 0 \end{cases} \quad -2n^2+(x-4)n+x^2-2x = 0;$$
$$x^2+(n-2)x-2n^2-4n = 0.$$
$$D = (n-2)^2-4\left(-2n^2-4n\right) = n^2-4n+4+8n^2+16n = 9n^2+12n+4 = (3n+2)^2;$$
$$x_1 = \frac{2-n-(3n+2)}{2} = \frac{2-n-3n-2}{2} = \frac{-4n}{2} = -2n;$$
$$x_2 = \frac{2-n+3n+2}{2} = \frac{2n+4}{2} = n+2.$$

Answer:
$$\begin{array}{l} x\neq 0, -2n\neq 0, n\neq 0, x_1 = -2n; x_2 = n+2; \\ x+2\neq 0, -2n+2\neq 0, n\neq 1, x_1 = -2n; x_2 = n+2; \\ 2-x\neq 0, 2+2n\neq 0, n\neq -1, x_1 = -2n; x_2 = n+2; \\ n = 0, x\in\varnothing. \end{array}$$

A lot of equations with parameters are offered to applicants to universities, which are very difficult to solve in the traditional way. The work is greatly facilitated by the technique, which consists in the fact that the parameter is considered an unknown variable, and the unknown is considered a coefficient. This technique is especially rational when the degree of the equation is higher than two..

$$x^4-2mx^2-x+m^2-m = 0.$$

Solution:

This is an equation of the fourth degree. Let's apply this technique:

$$x^4-2mx^2-m-x+m^2 = 0;$$
$$x^4-\left(2x^2+1\right)m-x+m^2 = 0;$$

Let's rewrite this equation as quadratic with respect to m:

$$m^2-\left(2x^2+1\right)m+x^4-x = 0;$$
$$D = \left(2x^2+1\right)^2-4\cdot 1\left(x^4-x\right) = 4x^4+4x^2+1-4x^4+4x = 4x^2+4x+1 = (2x+1)^2;$$
$$m_1 = \frac{2x^2+1-(2x+1)}{2} = \frac{2x^2+1-2x-1}{2} = \frac{2x^2-2x}{2} = x^2-x;$$

$m_2 = x^2+x+1$. Consider a set of equations:

$$\left[\begin{array}{l} x^2 - x = m, \\ x^2 + x + 1 = m \end{array}\right. \left[\begin{array}{l} x^2 - x - m = 0, \\ x^2 + x + 1 - m = 0 \end{array}\right. \begin{array}{l} D = 1 + 4m, \\ D = 1 - 4 + 4m = 4m - 3. \end{array}$$

$x_1 = \dfrac{1 - \sqrt{1+4m}}{2}$; $1 + 4m \geq 0,\ 4m \geq -1,\ m \geq -\dfrac{1}{4}$;

$x_2 = \dfrac{1 + \sqrt{1+4m}}{2}$; at $m \geq -\dfrac{1}{4}$;

$x_3 = \dfrac{-1 - \sqrt{4m-3}}{2}$; at $4m - 3 \geq 0, 4m \geq 3, m \geq \dfrac{3}{4}$;

$x_4 = \dfrac{-1 + \sqrt{4m-3}}{2}$; at $m \geq \dfrac{3}{4}$.

Answer: at $m \in \left(-\infty; -\dfrac{1}{4}\right)$ $x \in \varnothing$; at $m = -\dfrac{1}{4}$, $x = \dfrac{1}{2}$; at

$a \in \left(\dfrac{3}{4}; +\infty\right)$ $x = \dfrac{1 \pm \sqrt{1+4a}}{2}$; $x = \dfrac{-1 \pm \sqrt{4a-3}}{2}$; at $m = \dfrac{3}{4}$, $x = -\dfrac{1}{2}$;

at $m \in \left(-\dfrac{1}{4}; \dfrac{3}{4}\right)$ $x = \dfrac{1 \pm \sqrt{1+4m}}{2}$.

In some equations, it is advisable to introduce the parameter ourselves in order to facilitate the process of solving:

$(x^2 - 1)^2 + (\sqrt{3} + 1) \cdot (4x - \sqrt{3} - 1) = 0.$

Solution:

Converting the expression in the third parentheses:

$4x - \sqrt{3} - 1 = 4x - (\sqrt{3} + 1)$. Then the equation will have the form:

$(x^2 - 1)^2 + (\sqrt{3} + 1) \cdot (4x - (\sqrt{3} + 1)) = 0.$

Let's introduce the parameter $t = \sqrt{3} + 1$.

$(x^2 - 1)^2 + t \cdot (4x - t) = 0$; This is an equation of the fourth degree with respect to the variable x.

It is advisable to solve it in the previous way: t considered as a variable, and x as a coefficient.

$t \cdot (4x - t) + (x^2 - 1)^2 = 0;$

$4tx - t^2 + (x^2 - 1)^2 = 0 \,|\cdot (-1)$

$t^2 - 4xt - (x^2 - 1)^2 = 0;$

$D = (-4x)^2 + 4 \cdot 1 \cdot (x^2 - 1)^2 = 16x^2 + 4x^4 - 8x^2 + 4 = 4x^4 + 16x^2 + 4 = 4 \cdot (x^4 + 4x^2 + 1) = 4 \cdot (x^2 + 1)^2;$

$t_1 = \dfrac{4x - \sqrt{4(x^2+1)^2}}{2} = \dfrac{4x - 2(x^2+1)}{2} = 2x - (x^2 + 1);$

$t_2 = 2x + (x^2 + 1).$

Consider the set of equations:

$$\left[\begin{array}{l} 2x - x^2 - 1 = t, \\ 2x + x^2 + 1 = t \end{array}\right. \left[\begin{array}{l} -x^2 + 2x - 1 - t = 0 \,|\cdot (-1), \\ x^2 + 2x + 1 - t = 0 \end{array}\right. \left[\begin{array}{l} x^2 - 2x + 1 + t = 0, \\ x^2 + 2x + 1 - t = 0. \end{array}\right.$$

Considering that $t = \sqrt{3} + 1$, we have a new set of equations:

$$\left[\begin{array}{l} x^2-2x+1+\sqrt{3}+1=0, \\ x^2+2x+1-\sqrt{3}-1=0. \end{array}\right. \quad \left[\begin{array}{l} x^2-2x+\sqrt{3}+2=0, \\ x^2+2x-\sqrt{3}=0. \end{array}\right.$$

$D=(-2)^2-4\left(\sqrt{3}+2\right)=4-4\sqrt{3}-8=-4\sqrt{3}-4<0,\ x\in\varnothing.$

$D=4-4\cdot\left(-\sqrt{3}\right)=4+4\sqrt{3}=4\cdot\left(1+\sqrt{3}\right)>0,$

$$x_1=\frac{-2-\sqrt{4\left(1+\sqrt{3}\right)}}{2}=\frac{-2-2\sqrt{1+\sqrt{3}}}{2}=-1-\sqrt{1+\sqrt{3}};$$

$x_2=-1+\sqrt{1+\sqrt{3}}.$

Answer: $-1-\sqrt{1+\sqrt{3}};\ -1+\sqrt{1+\sqrt{3}}.$

Self-study assignments:

$b\cdot x=0.$ Answer: at $b\neq 0,\ x=0.$

$c\cdot x=c.$ Answer: at $c=0,\ x\in R;$ at $c\neq 0,\ x=1.$

$x-ax=-2.$ Answer: at $a=1,\ x\in\varnothing;$ at $a\neq 1,\ x=\dfrac{2}{a-1}.$

$\left(a^2-1\right)x=2a^2+a-3.$

Answer: at $a=1,\ x\in R;$ at $a=-1,\ x\in\varnothing;$ at $a\neq 1,\ x=\dfrac{2a+3}{a+1}.$

$ax=x+3.$ Answer: at $a\neq 1,\ x=\dfrac{3}{a-1};$ at $a=1,\ x\in\varnothing.$

$4+ax=3x+1.$ Answer: at $a\neq 3,\ x=\dfrac{3}{3-a};$ at $a=3,\ x\in\varnothing.$

$\dfrac{10}{5x-m}=\dfrac{3}{mx-2}.$

Answer: at $m=1{,}5$ and $m=\pm\sqrt{10}$ $x\in\varnothing;$ at $m\neq 1{,}5$ and $m\neq\pm\sqrt{10}$ $x=\dfrac{20-3m}{10m-15}.$

$\dfrac{mx-4}{3x-m}=\dfrac{3}{5}.$

Answer: at $m \neq 1{,}8$ and $m \neq \pm\sqrt{18}$ $x = \dfrac{2x-3m}{5m-9}$; at $m = 1{,}8$ and $m = \pm\sqrt{18}$ $x \in \varnothing$.

$$\frac{x+a}{x+b} - \frac{x+b}{x+a} + \frac{4ab}{a^2-b^2} = 0.$$

Answer: at $|a| \neq |b|$, $a \neq 0$, $b \neq 0$, then $x = -\dfrac{a^2+b^2}{2a}$; $x = -\dfrac{a^2+b^2}{2b}$.

If $a = 0$, $b \neq 0$, then $x = -2a$;

If $b = a \neq 0$, then $x = 0$;

If $b = 0$, or $a = 0$, then $x \in \varnothing$.

$$\frac{1}{2n+nx} - \frac{1}{2x-x^2} = \frac{2(n+3)}{x^3-4x}.$$ Answer: at $n = 0$, $x \in \varnothing$.

at $n \neq 0$, $x_1 = -2n; x_2 = n+2$;

at $n \neq 1$, $x_1 = -2n; x_2 = n+2$;

at $n \neq -1$, $x_1 = -2n; x_2 = n+2$.

$$x^3 + 2ax^2 + a^2x + a - 1 = 0.$$

Answer: at $a \in (-3;1)$ $x \in \varnothing$; at $a = -3$ and $a = 1$ $x = \dfrac{-1-a}{2}$; at $a \in (-\infty;-3) \cup (1;+\infty)$

$x = -1 - a \pm \sqrt{a^2+2a-3}$.

$$a = \frac{1}{a} + \frac{a-1}{a(x-1)}.$$

Answer: at $a \neq \pm 1$ and $a \neq 0$ $x = \dfrac{a+2}{a+1}$; at $a = 1$ $x \in (-\infty;1) \cup (1;+\infty)$; at $a = -1$ and

$a = 0$ $x \in \varnothing$.

$\dfrac{2(a+1)x}{a} = \dfrac{7}{a} + 3(x+1)$. Answer: at $a \neq 2$ and $a \neq 0$ $x = \dfrac{7+3a}{2-a}$; at $a = 0$ and $a = 2$ $x \in \varnothing$.

$$\frac{a+3}{a+2} = \frac{2}{x} - \frac{5}{(a+2)x}.$$

Answer: at $a \neq -3$ and $a \neq -2$ and $a \neq 0{,}5$ $x = \dfrac{2a-1}{a+3}$; at $a = -3$, $a = -2$, $a = 0{,}5$ $x \in \varnothing$.

$\frac{1+x}{1-x}=\frac{a}{c}$. Answer: at $a+c\neq 0$ and $c\neq 0$ $x=\frac{a-c}{a+c}$; at $a=-c,\ c=0$ $x\in\varnothing$.

$\frac{5}{ax-4}=\frac{1}{9x-a}$. Answer: $x=\frac{5a-4}{45-a}$ at $a\pm 45$ and $a\neq\pm 6$; at $a=4{,}5$ and $a=\pm 6$ $x\in\varnothing$.

$\frac{1}{x-1}+\frac{1}{x-6}=1+\frac{1}{6}$.

Answer: $x_1=b+1,\ x_2=\frac{2b}{b+1}$ at $b\neq -1,\ b\neq 0,\ b\neq 1;\ x\in\varnothing$ at $b=0$.

$\frac{1}{2n+nx}-\frac{1}{2x-x^2}=\frac{2(n+3)}{x^3-4x}$.

Answer:
$$\begin{array}{l} x\neq 0, -2n\neq 0, n\neq 0, x_1=-2n; x_2=n+2;\\ x+2\neq 0, -2n+2\neq 0, n\neq 1, x_1=-2n; x_2=n+2;\\ 2-x\neq 0, 2+2n\neq 0, n\neq -1, x_1=-2n; x_2=n+2;\\ n=0, x\in\varnothing. \end{array}$$