Section 5
Rational equations

Let us recall some of the basic mathematical concepts for this section..

A function is a correspondence between two sets *X* and *Y*, in which each element of the set *X* corresponds to only one element of the set *Y*.

The fact that a given function is written as: y = f(x), x – argument, y – function, f – conformity law.

The domain of the function - is the set of values of the argument for which the function is meaningful. The area of definition is indicated by *Д(y)*.

$y = 8x^3 + 10x^2 - 12x + 14, \qquad Д(y) = (-\infty;\ +\infty).$

The value of the argument at which the value of the function is equal to zero is called the root, or zero of the function. Function $y = (x+2)\cdot(x+1)\cdot x(x-1)\cdot(x-2)$ has five zeros or five roots: $x = -2; \quad x = -1; \quad x = 0; \quad x = 1; \quad x = 2.$

Function $f(x)$ is called paired if for all values of the argument *x* belonging to its domain of definition, the equality: $f(-x) = f(x).$

Function $f(x) = x^4 + 3x^2 - 8 -$ paired function, because

$f(-x) = (-x)^4 + 3(-x)^2 - 8 = x^4 + 3x^2 - 8 = f(x),$

$f(-x) = f(x).$

The graph of an even function is symmetric about the ordinate axis.

Function $f(x)$ is called odd if for all values of the argument *x*, belonging to its domain of definition, the equality: $f(-x) = -f(x).$

The plot of an odd function is symmetric about the origin.

Functions can be even, odd and general, that is, those that do not belong to either even or odd.

Functions are classified into two large groups: algebraic and non-algebraic, that is, transcendental.

A function is called algebraic if a finite number of algebraic operations are performed on its argument (addition, subtraction, multiplication, division, exponentiation and extraction of a root with a natural exponent.

Equality with the variable whose value you want to find is called the equation.

Algebraic equations are divided into rational and irrational.

Rational algebraic equations are those in which algebraic operations are performed on the argument, except for the extraction of the root. All other algebraic equations are called irrational.

For example: $\dfrac{4}{x-2} + \dfrac{x}{x^2-1} = 5x + 6 -$ rational equation, and the equation

$\sqrt[4]{x-6} + \sqrt[4]{x+6} = \dfrac{4}{\sqrt{x}}$ is irrational. The value of the argument, for which equal value is transformed into the correct numerical equality, is called the root of the equation.

A whole rational equation is an equation of the form $F(x)=0$, де $F(x)-$ whole rational degree function *n*.

When choosing equations for solving, the dominant place should be given to those of them that are solved after school hours and during the period of pre-university training, from those sections of elementary mathematics, in-depth study, which is supposed to be a program in mathematics for students of higher educational institutions.

We will show several ways to solve entire rational equations of higher second degree. If the equation is reduced and its terms are arranged in descending degrees, then it can be solved in this way: $x^4+8x^3+10x^2-24x+5=0.$

Solution:

1). We are looking for the divisors of the free term of the equation, that is, the numbers 5. This: –1; 1; –5; 5.

2). The one of these four numbers that turns the left side of the equation to zero, we count it as the root. This is the number 1, since

$1^4+8\cdot 1^3+10\cdot 1^2-24\cdot 1+5=1+8+10-24+5=24-24=0.$

3). We divide the left side of this equation into a binomial $x-1$:

$$
\begin{array}{l|l}
x^4+8x^3+10x^2-24x+5 & x-1 \\
\cline{2-2}
-\underline{x^4-x^3} & x^3+9x^2+19x-5 \\
\quad -\begin{array}{l} 9x^3+10x^2 \\ \underline{9x^3-9x^2} \end{array} & \\
\qquad -\begin{array}{l} 19x^2-24x \\ \underline{19x^2-19x} \end{array} & \\
\qquad\quad -\begin{array}{l} -5x+5 \\ \underline{-5x+5} \end{array} & \\
\qquad\qquad\quad 0. &
\end{array}
$$

4). Looking for whole roots of the equation $x^3+9x^2+19x-5=0$ the previous way. $\pm 1;\ \ \pm 5-$ free term divisors and only number –5 is the root of this equation, therefore:

$$
\begin{array}{l|l}
x^3+9x^2+19x-5 & x+5 \\
\cline{2-2}
-\underline{x^3-5x^2} & x^2+4x-1 \\
\quad -\begin{array}{l} 4x^2+19x \\ \underline{4x^2+20x} \end{array} & \\
\qquad -\begin{array}{l} -x-5 \\ \underline{-x-5} \end{array} & \\
\qquad\qquad 0. &
\end{array}
$$

5). We solve the quadratic equation $x^2+4x-1=0.$

$Д = 4^2 + 4 = 20 > 0,$

$x_1 = \dfrac{-4-\sqrt{20}}{2} = \dfrac{-4-2\sqrt{5}}{2} = -2-\sqrt{5},$

$x_2 = -2+\sqrt{5}.$

Answer: $-5;\ -2-\sqrt{5};\ -2+\sqrt{5};\ 1.$

This method is suitable when at least one of the roots of the equation is a divisor of its free term and the first coefficient is equal to one. If the first coefficient of the equation is different from unity, then sometimes you can use this technique.

Find the rational roots of the equation: $4x^4 + 8x^3 + x^2 + 3x + 9 = 0.$

Solution:

$4x^4 + 8x^3 + x^2 + 3x + 9 = 0 \mid \cdot 2^2;$

$2^4 \cdot x^4 + 4 \cdot 2^3 \cdot x^3 + 2^2 \cdot x^2 + 12x + 36 = 0;$ Introducing a replacement $2x = y,$ then

$(2x)^4 + 4\cdot(2x)^3 + (2x)^2 + 6\cdot(2x) + 36 = 0;$

the equation becomes: $y^4 + 4y^3 + y^2 + 6y + 36 = 0.$

Let's use the previous method: $36 M\{\pm 1;\ \pm 2;\ \pm 3;\ \pm 4;\ \pm 6;\ \pm 9;\ \pm 12;\ \pm 18;\ \pm 36.\}$

Since all the coefficients of this equation are positive, the roots must be sought among the negative divisors of the number 36, that is, among the numbers: −1; −2; −3; −4; −6;−9;−12;−18;−36.

We denote $y^4 + 4y^3 + y^2 + 6y + 36 = F(y.)$

Testing negative divisors 36: $F(-1) = (-1)^4 + 4\cdot(-1)^3 + (-1)^2 + 6\cdot(-1) + 36 \neq 0,$

it means that *y*= *–1* – is not a root of the equation.

$F(-2) = (-2)^4 + 4\cdot(-2)^3 + (-2)^2 + 6\cdot(-2) + 36 = 16 - 32 + 4 - 12 + 36 \neq 0,$

it means that *y*= *–2* – is not a root of the equation.

$F(-3) = (-3)^4 + 4\cdot(-3)^3 + (-3)^2 + 6\cdot(-3) + 36 = 81 - 108 + 9 - 18 + 36 = 126 - 126 = 0,$

it means that *y*= *–3* – is the root of the equation.

$$
\begin{array}{l|l}
y^4 + 4y^3 + y^2 + 6y + 36 & y+3 \\ \cline{2-2}
-\underline{y^4 + 3y^3} & y^3 + y^2 - 2y + 12 \\
\quad -\ y^3 + y^2 & \\
\quad\ \ \underline{y^3 + 3y^2} & \\
\qquad -\ -2y^2 + 6y & \\
\qquad\ \ \ \underline{-2y^2 - 6y} & \\
\qquad\qquad -\ 12y + 36 & \\
\qquad\qquad\ \ \ \underline{12y + 36} & \\
\qquad\qquad\qquad 0. &
\end{array}
$$

Returning to the replacement *y=2x*, find $x = \dfrac{y}{2};\ \ x = -\dfrac{3}{2} = -1{,}5 -$ root equations. The equation $y^3 + y^2 - 2y + 12 = 0$ has no whole roots when none of the divisors of 12 turns it into a true numerical equality.

Answer: –1,5.

Solve the equation $4x^3+5x-3=0$;

Solution:

$4x^3+5x-3=0 \,|\times 2$

$8x^3+10x-6=0;$

$25x^3+x-26=0 \,|\cdot 5$

$125x^3+5x-130=0$

$(5x)^3+5x-130=0$

Replacement $5x=y.$

$y^3+y-130=0$

$y=5-$ the root of the equation, since $5^3+5-130=0$.

$2^3x^3+5\cdot 2x-6=0;$

$(2x)^3+5\cdot 2x-6=0;$ Replacement $2x=y$;

Free member divisors:

$\pm 1;\ \pm 2;\ \pm 3;\ \pm 6.$ We are convinced by direct test that

$y=1-$ the root of this equation, because $1^3+5\cdot 1-6=0.$ Then:

$$
\begin{array}{r|l}
y^3+5y-6 & y-1 \\
\hline
-\underline{y^3-y^2} & y^2+y+6 \\
y^2+5y & \\
-\underline{y^2-y} & \\
6y-6 & \\
-\underline{6y-6} & \\
0 &
\end{array}
$$

$y^2+y+6=0-$ has no real roots, since $D=1-24=-23<0$.

Well then $2x=1;\quad x=0{,}5.$

Answer: 0,5.

$y^3+5y-6=0;$

$$
\begin{array}{r|l}
y^3+y-130 & y-5 \\
\hline
-\underline{y^3-5y^2} & y^2+5y+26 \\
5y^2+y & \\
-\underline{5y^2-25y} & \\
26y-130 & \\
-\underline{26y-130} & \\
0. &
\end{array}
$$

The equation $y^2+5y+26=0-$ has no real roots, because $D=25-104<0.$

Well then, $5x=5,\quad x=1.$

Answer: 1

From the set of rational equations, it is advisable to single out symmetric equations, that is, of this form:

$$a_0\cdot x^n+a_1\cdot x^{n-1}+a_2\cdot x^{n-2}+\ldots+a_2x^2+a_1x+a_0=0.$$

When solving symmetric equations, the following sequence should be followed:

1). If the exponent of the equation is even, that is, $n=2k,$ then when divided by x^k, that is on x to a power half as much as the exponent of the equation, we obtain an equation equivalent to the given one, which is subsequently solved by the substitution method.

2) if the exponent of the equation is not even, that is $n = 2k + 1$ first you need to find at least one root by direct test x_1.
Then solve this equation as the previous ones.
$12x^5 + 8x^4 - 45x^3 - 45x^2 + 8x + 12 = 0.$

Solution:

This is a symmetric equation with an odd exponent, and by direct testing we come to the conclusion that $x = -1 -$ one of the roots of this equation →
$12 \cdot (-1)^5 + 8 \cdot (-1)^4 - 45 \cdot (-1)^3 - 45 \cdot (-1)^2 + 8 \cdot (-1) + 12 = 0.$

$$\begin{array}{r|l} 12x^5 + 8x^4 - 45x^3 - 45x^2 + 8x + 12 & x+1 \\ \hline -\,12x^5 + 12x^4 & 12x^4 - 4x^3 - 41x^2 - 4x + 12 \\ -\,{-4x^4 - 45x^3} & \\ -4x^4 - 4x^3 & \\ -\,{-41x^3 - 45x^2} & \\ -41^3 - 41x^2 & \\ -\,{-4x^2 + 8x} & \\ -4x^2 - 4x & \\ -\,{12x + 12} & \\ 12x + 12 & \\ 0. & \end{array}$$

$$12x^4 - 4x^3 - 41x^2 - 4x + 12 = 0 \,|: x^2;$$

$$12x^2 - 4x - 41 - \frac{4}{x} + \frac{12}{x^2} = 0;$$

$$\left(12x^2 + \frac{12}{x^2}\right) + \left(-4x - \frac{4}{x}\right) - 41 = 0;$$

$$12 \cdot \left(x^2 + \frac{1}{x^2}\right) - 4 \cdot \left(x + \frac{1}{x}\right) - 41 = 0;$$

We denote $x + \dfrac{1}{x} = t.$ Let us square both sides of this equality:

$\left(x + \dfrac{1}{x}\right)^2 = t^2;$ $\quad x^2 + 2 \cdot x \cdot \dfrac{1}{x} + \dfrac{1}{x^2} = t^2;$ $\quad x^2 + \dfrac{1}{x^2} = t^2 - 2,$ then the equation will have the form: $12 \cdot (t^2 - 2) - 4t - 41 = 0;$ $\quad 12t^2 - 24 - 4t - 41 = 0;$ $\quad 12t^2 - 4t - 65 = 0;$

$D = 4^2 - 12 \cdot 4 \cdot (-65) = 16 + 3120 = 3136 = 56^2;$ $\quad t_1 = \dfrac{4 - 56}{24} = -\dfrac{52}{24} = \dfrac{13}{6};$ $\quad t_2 = \dfrac{4 + 56}{24} = \dfrac{5}{2}.$

Let us solve such a set of equations:

$$\left[\begin{array}{l} x + \dfrac{1}{x} = -\dfrac{13}{6}, \\ x + \dfrac{1}{x} = \dfrac{5}{2}. \end{array}\right. \qquad \left[\begin{array}{l} \begin{cases} x^2 + \dfrac{13}{6}x + 1 = 0 \,|\times 6, \\ x \neq 0. \end{cases} \\ \begin{cases} x^2 - \dfrac{5}{2}x + 1 = 0, |\times 2, \\ x \neq 0. \end{cases} \end{array}\right. \qquad \left[\begin{array}{l} \begin{cases} 6x^2 + 13x + 6 = 0, \\ x \neq 0. \end{cases} \\ \begin{cases} 2x^2 - 15x + 2 = 0, \\ x \neq 0. \end{cases} \end{array}\right.$$

We solve each equation totality.

$D = 13^2 - 4 \cdot 6 \cdot 6 = 169 - 144 = 25 = 5^2;$ $\qquad D = 5^2 - 4 \cdot 2 \cdot 2 = 25 - 16 = 9 = 3^2;$

$x_1 = \dfrac{-13 - 5}{12} = -\dfrac{18}{12} = -\dfrac{3}{2};$ $\qquad x_3 = \dfrac{5 - 3}{4} = \dfrac{1}{2};$

$x_2 = \dfrac{-13 + 5}{12} = -\dfrac{8}{12} = -\dfrac{2}{3};$ $\qquad x_4 = \dfrac{5 + 3}{4} = 2;$ $\qquad x_5 = -1.$

Answer: $-\frac{3}{2}$; $-\frac{2}{3}$; $\frac{1}{2}$; 2; -1.

Self-study assignments:

Solve the equation:

5.1. $3x+12=0;$ Answer: $-4.$

5.2. $\frac{1}{2}x-5=0;$ Answer: $10.$

5.3. $2x+6-5x=7+4\cdot(2-3x);$ Answer: $1.$

5.4. $\frac{3x-2}{2}-\frac{8-x}{3}=\frac{2\cdot(1{,}25x+5)}{5}-2x+1;$ Answer: $2.$

5.5. $6x^2=0;$ Answer: $0.$

5.6. $9x^2=81;$ Answer: $-3;\ 3.$

5.7. $\frac{x^2}{27}=3;$ Answer: $-9;\ 9.$

5.8. $x^2-36=0;$ Answer: $-6;\ 6.$

5.9. $5-5x^2=0;$ Answer: $-1;\ 1.$

5.10. $2x^2+32=0;$ Answer: $\varnothing.$

5.11. $x^2-9x=0;$ Answer: $0;\ 9.$

5.12. $2x^2+7x=0;$ Answer: $0;\ -3{,}5.$

5.13. $x^2+8x+16=0;$ Answer: $-4.$

5.14. $7x-12-x^2=0;$ Answer: $3;\ 4.$

5.15. $3x^2-4x-4=0;$ Answer: $-\frac{2}{3};\ 2.$

5.16. $6x^3+7x^2-x-2=0;$ Answer: $-1;\ -\frac{2}{3};\ \frac{1}{2}.$

5.17. $3x^4-2x^3+2x^2-2x-1=0;$ Answer: $-\frac{1}{3};\ 1.$

5.18. $2x^4+10x^3 10x^2+6x+36=0;$ Answer: $-3.$

5.19. $3x^4-12{,}5x^3-13{,}5x^2+4{,}5x+2{,}5=0;$ Answer: $-1;\ \frac{1}{3};\ \frac{1}{2};\ 5.$

5.20. $x^5-4x^4-11x^3-24=6x^2+4x;$ Answer: $-2;\ 1;\ 6.$

5.21. $6x^2+4x-1=22x^3+17x^4-6x^5;$ Answer: $-1;\ \frac{1}{2};\ \frac{1}{3};\ 2+\sqrt{3};\ 2-\sqrt{3}.$

5.22. $4x^4-16x^3 3x^2+4x-1=0;$ Answer: $-\frac{1}{2};\ \frac{1}{2};\ 2+\sqrt{3};\ 2-\sqrt{3}.$

5.23. $6-35x+62x^2=35x^3-6x^4;$ Answer: $-2;\ \frac{1}{2};\ 3;\ \frac{1}{3}.$

5.24. $2-x^2+9x=-2x^4-9x^3;$ Answer: $\frac{-5-\sqrt{21}}{2};\ \frac{-5+\sqrt{21}}{2}.$

There are many equations for which the most rational solution is to introduce a new variable. In this case, it is necessary to include as many members as possible in the substitution.

$\left(2x^2+x+3\right)^2+2\cdot\left(2x^2+x-2\right)-14=0.$

Solution:

Substitution $2x^2+x+3=t,$ then $2x^2+x=t-3.$ This equation will have the form:

$t^2+2\cdot(t-3-2)-14=0;\quad t^2+2\cdot(t-5)-14=0;\quad t^2+2t-10-14=0;\quad t^2+2t-24=0.$

By Vieta's theorem: $t_1=-6;\quad t_2=4.$ Let us solve such a set of equations:

$$\left[\begin{array}{l}2x^2+x+3=-6,\\2x^2+x+3=4,\end{array}\right.\qquad \left[\begin{array}{l}2x^2+x+9=0,\\2x^2+x-1=0,\end{array}\right.\qquad \left[\begin{array}{l}D=1-72<0,\quad x\in\varnothing,\\D=1+8=9>0,\quad x_1=\dfrac{-1-3}{4}=-1,\\x_2=\dfrac{-1+3}{4}=\dfrac{1}{2}.\end{array}\right.$$

Answer: $-1;\quad \frac{1}{2}.$

Solve the equation: $(2x-1)\cdot(2x+3)\cdot(x+2)\cdot(x+4)=-9$.

Solution:

Let's multiply the first and fourth and second and third factors:

$\left(2x^2+8x-x-4\right)\cdot\left(2x^2+4x+3x+6\right)=-9;\quad \left(2x^2+7x-4\right)\cdot\left(2x^2+7x+6\right)+9=0;$

$2x^2+7x+6=t;\quad 2x^2+7x=t-6.\quad (t-6-4)\cdot t+9=0;\quad (10-t)\cdot t+9=0;$

$t^2-10t+9=0;\quad t_1=1;\quad t_2=9.$

$$\left[\begin{array}{l}2x^2+7x+6=1,\\2x^2+7x+6=9.\end{array}\right.\qquad \left[\begin{array}{l}2x^2+7x+5=0,\\2x^2+7x-3=0.\end{array}\right.\qquad \left[\begin{array}{l}D=49-40=9,\quad x_1=\dfrac{-7-3}{4}=-2{,}5;\quad x_2=\dfrac{-7+3}{4}=-\\D=49+24=73,\quad x_3=\dfrac{-7+\sqrt{73}}{4};\quad x_4=\dfrac{-7-\sqrt{73}}{4}.\end{array}\right.$$

Answer: $-2{,}5;\quad -1;\quad \frac{-7-\sqrt{73}}{4};\quad \frac{-7+\sqrt{73}}{4}.$

Let's solve a more complicated equation: $\left(x^2+3x-1\right)^2+3x^2\cdot\left(x^2+3x-1\right)=4x^4.$

Solution:

Direct substitution shows that $x=0-$ is not the root of this equation, and therefore $x^4\neq 0,$ we divide both sides of the equation by

$x^4:\quad \dfrac{\left(x^2+3x-1\right)^2}{x^4}+3x^2\cdot\dfrac{x^2+3x-1}{x^4}=\dfrac{4x^4}{x^4};\quad \left(\dfrac{x^2+3x-1}{x^2}\right)^2+3\cdot\dfrac{x^2+3x-1}{x^2}-4=0.$

Substitution $\dfrac{x^2+3x-1}{x^2}=t,$ then the equation takes the form: $t^2+3t-4=0,$

$t_1=-4;\quad t_2=1.$ Then we solve the set of equations:

$$\left[\begin{array}{l}\frac{x^2+3x-1}{x^2}=-4,\\ \frac{x^2+3x-1}{x^2}=1.\end{array}\right. \quad \left[\begin{array}{l}\frac{x^2+3x-1}{x^2}+4=0\Big|\times x^2,\\ \frac{x^2+3x-1}{x^2}-1=0\Big|\times x^2.\end{array}\right. \quad \left[\begin{array}{l}x^2+3x-1+4x^2=0,\\ x^2+3x-1-x^2=0.\end{array}\right. \quad \left[\begin{array}{l}5x^2+3x-1=0, \quad \left[\begin{array}{l}D=9+20=29,\\ x_1=\frac{-3-\sqrt{29}}{10},\ x_2=\frac{-3+\sqrt{29}}{10}.\end{array}\right.\\ 3x-1=0. \quad x_3=\frac{1}{3}.\end{array}\right.$$

Answer: $x_1=\frac{-3-\sqrt{29}}{10}$, $x_2=\frac{-3+\sqrt{29}}{10}$, $x_3=\frac{1}{3}$.

In equations of higher degrees of the form $x^4+(x+8)^4=100$ replacement is made somewhat unusual.

Solution:

$(x+0)^4+(x+8)^4=100$. Looking for the arithmetic mean of numbers 0 and 8: $\frac{0+8}{2}=4$. Replacement $x=t-4$. Then the equation is reduced to the form:

$(t-4)^4+(t-4+8)^4=100;\quad (t-4)^4+(t+4)^4=100;\quad \left((t-4)^2\right)^2+\left((t+4)^2\right)^2=100;$

$\left(t^2-8t+16\right)^2+\left(t^2+8t+16\right)^2=100;$

$t^4+64t^2+256-16t^3+32t^2-256t+t^4+64t^2+256+16t^3+32t^2+256t=100;$

$2t^4+192t^2+512-100=0\big|:2;\qquad t^4+96t^2+206=0.$

We introduce a new variable $t^2=z$, then

$z^2+96z+206=0;\quad D=96^2-4\cdot 206=9216-824=8392;$

$Z=\frac{-96-\sqrt{8392}}{2}=-48-\sqrt{2098};\qquad Z_2=-48+\sqrt{2098}.$

Let's solve the set of equations: $\left[\begin{array}{l}t^2=-48-\sqrt{2098},\\ t^2=-48+\sqrt{2098}\end{array}\right. \quad \left[\begin{array}{l}t\in\varnothing,\ \text{в множині R},\\ |48|>\sqrt{2098},\ \text{а тому } t\in\varnothing.\end{array}\right.$

Answer: the equation has no real roots.

Solving equations of the type: $x^4+4x-1=0$ is reduced to the addition and subtraction of expressions that will make it possible to select the squares of binomials with the subsequent application of the formula for the difference of squares.

Solution:

$x^4+2x^2+1-2x^2+4x-1-1=0;\quad \left(x^4+2x^2+1\right)-2\cdot\left(x^2-2x+1\right)=0;\quad \left(x^2+1\right)^2-\left(\sqrt{2}\right)^2\cdot(x-1)^2=0;$

$\left(x^2+1-\sqrt{2}\cdot(x-1)\right)\cdot\left(x^2+1+\sqrt{2}\cdot(x-1)\right)=0.$

$$\left[\begin{array}{l}x^2+1-\sqrt{2}x+\sqrt{2}=0,\\ x^2+1+\sqrt{2}x-\sqrt{2}=0\end{array}\right. \quad \left[\begin{array}{l}x^2-\sqrt{2}x+\sqrt{2}+1=0,\quad D=2-4\sqrt{2}-4<0,\ x\in\varnothing;\\ x^2+\sqrt{2}x+1-\sqrt{2}=0,\quad D=2+4\sqrt{2}-4=4\sqrt{2}-2>0.\end{array}\right.$$

$x_1=\frac{-\sqrt{2}-\sqrt{4\sqrt{2}-2}}{2};\quad x_2=\frac{-\sqrt{2}+\sqrt{4\sqrt{2}-2}}{2}.$

Answer: $\frac{-\sqrt{2}-\sqrt{4\sqrt{2}-2}}{2}$, $\frac{-\sqrt{2}+\sqrt{4\sqrt{2}-2}}{2}$.

Self-study assignments:

Solve Equations:

5.25. $x^3+9x^2+11x+9=30.$ Answer: $-7;\ -3;\ 1.$

5.26. $x^4-x^3-2\cdot\left(3x^2-2x-4\right)=0.$ Answer: $-2;\ -1;\ 2;\ 1.$

5.27. $3\cdot\left(3x^3-5x^2-4\right)=32x.$ Answer: $-\frac{2}{3};\ \frac{2}{3};\ 3.$

5.28. $9x^2+5x=-4\cdot\left(x^4+2x^3+0{,}25\right).$ Answer: $-\frac{1}{2}.$

5.29. $(x+4)\cdot(x+7)\cdot(x+1)\cdot(x-2)=19.$ Answer: $\frac{-5-\sqrt{5}}{2};\ \frac{-5+\sqrt{5}}{2};\ \frac{-5-\sqrt{85}}{2};\ \frac{-5+\sqrt{85}}{2}.$

5.30. $\left(x^2+x+4\right)^2+8x\cdot\left(x^2+x+4\right)=-15x^2.$ Answer: $-2;\ -3-\sqrt{5};\ -3+\sqrt{5}.$

5.31. $27x^3+8=(2x+3)^3+(x-1)^3.$ Answer: $3;\ -\frac{1}{2};\ -\frac{2}{3}.$

5.32. $x^4+4x=1.$ Answer: $\frac{1}{2}\cdot\left(-\sqrt{2}-\sqrt{4\sqrt{2}-2}\right);\ \frac{1}{2}\cdot\left(-\sqrt{2}+\sqrt{4\sqrt{2}-2}\right).$

5.33. $(x+3)^4+(x+5)^4=16.$ Answer: $-3;\ -5.$

5.34. $(x-2)^6-(x-4)^6=64.$ Answer: $2;\ 4;\ -\frac{1}{2};\ \frac{1}{2};\ 2-\sqrt{3};\ 2+\sqrt{3}.$

5.35. $4x^4-16x^3+3x^2+4x-1=0.$ $x^2+\frac{81x^2}{(9+x)^2}=40.$ Answer: $\frac{2-\sqrt{19}}{2};\ \frac{2+\sqrt{19}}{2}.$

5.35. $(x-1)^5+(x-3)^5=242\cdot(x+1).$ Answer: $-2;\ -1;\ 0.$

Equations of the form $\frac{P_1(x)}{G_1(x)}+\frac{P_2(x)}{G_2(x)}+\ldots+\frac{P_m(x)}{G_m(x)}=0,$ где $\mathrm{P}_1(x), P_2(x),\ldots$

$P_m(x),\ G_1(x),\ G_2(x),\ldots$

$G_m(x)-$ entire rational functions, it is called a rational equation. The classical scheme for solving rational equations is as follows:

1) the right side of the equation is turned to zero;
2) on the left side reduce all fractions to a common denominator;
3) after reducing such terms, the condition of equality of the fraction to zero is used, that is, a system is formed from the equation and the inequality;
4) check whether the roots of the resulting equation have been found that satisfy the inequality of the system.

For example $\frac{x-1}{x+2}+\frac{2x+8}{4-x^2}=-\frac{x+1}{2-x}.$

Solution:

$$\frac{x-1}{x+2}+\frac{x+1}{2-x}+\frac{2x+8}{4-x^2}=0;\quad \frac{2x-x^2-2+x+2x+x^2+2+x+2x+8}{(2-x)\cdot(2+x)}=0;\quad \frac{8x+8}{(2-x)\cdot(2+x)}=0;$$

$$-\begin{cases}8x+8=0,\\(2-x)\cdot(2+x)\neq 0,\end{cases}\quad \begin{cases}8x=-8x,\\x\in(-\infty;\,-2)\cup(-2;\,2)\cup(2;\,+\infty),\end{cases}\quad x=-1,\ -1\in(-2;\,2).$$

Answer: $-1.$

In our opinion, this method of solving rational equations rather leads to the goal:

1) establish R.O.V.V. equations;

2) both sides of the equation are multiplied by the common denominator of the fractions of the equation;

3) solve the whole rational equation;

4) check the belonging of the found roots R.O.V.V.

Solve the equation: $\frac{2}{x^2+5x}+\frac{3}{2x-10}=\frac{15}{x^2-25}$.

Solution:

We find R.O.V.V. equations:

$$\left[\begin{array}{l} x^2+5x\neq 0,\\ 2x-10\neq 0,\\ x^2-25\neq 0, \end{array}\right.\quad \left[\begin{array}{l} x(x+5)\neq 0,\\ 2x\neq 10,\\ (x-5)\cdot(x+5)\neq 0, \end{array}\right.\quad \left[\begin{array}{l} x\neq 0,\\ x\neq -5,\\ x\neq 5,\\ x\neq 5,\\ x\neq -5 \end{array}\right.\quad \left[\begin{array}{l} x\neq 0,\\ x\neq -5,\\ x\neq 5. \end{array}\right.$$

R.O.V.V. $(-\infty;\ -5)\cup(-5;\ 0)\cup(0;\ 5)\cup(5;\ +\infty)$.

Common denominator of fractions $2\cdot x\cdot(x+5)\cdot(x-5)$.

We multiply both sides of the equation by a common denominator:

$$\frac{2}{x(x+5)}+\frac{3}{2(x-5)}=\frac{15}{(x-5)(x+5)}\Big|\cdot 2x(x+5)(x-5),\quad 4\cdot(x-5)+3x(x+5)=15\cdot 2x;$$

$$4x-20+3x^2+15x=30x;\quad 3x^2-11x-20=0;\quad D=121+240=361=19^2;$$

$$x_1=\frac{11-19}{6}=-\frac{8}{6}=-\frac{4}{3};\quad x_2=\frac{11+19}{6}=5;\quad 5\notin О.Д.З.\quad -\frac{4}{3}\in О.Д.З.$$ Answer: $-\frac{4}{3}$.

To go to a whole rational equation, this method makes it possible:

1) find R.O.V.V. equations;
2) reduce the harder part of the equation to a common denominator;
3) using the main property of proportion, we obtain the whole rational equation.

Solve the equation $\frac{1}{x-4}-\frac{1}{x-2}=\frac{1}{4}$.

Solution:

Reducing the left side of the equation to a common denominator, we find R.O.V.V.

$$\frac{1\cdot(x-2)-1\cdot(x-4)}{(x-2)\cdot(x-4)}=\frac{1}{4},\qquad \left[\begin{array}{l} x-4\neq 0,\\ x-2\neq 0 \end{array}\right.\quad \left[\begin{array}{l} x\neq 4,\\ x\neq 2 \end{array}\right.$$

R.O.V.V.: $(-\infty;\ 2)\cup(2;\ 4)\cup(4;\ +\infty)$. $\frac{x-2-x+4}{(x-4)\cdot(x-2)}=\frac{1}{4};\quad \frac{2}{(x-4)\cdot(x-2)}=\frac{1}{4}$.

By the main property of proportion, we have $(x-4)\cdot(x-2)\cdot 1=2\cdot 4$,

$x^2-2x-4x+8-8=0,\quad x^2-6x=0,\quad x_1=0,\quad x_2=6.$ The numbers 0 and 6 belong to R.O.V.V.

Answer: 0; 6.

There are such rational equations, for the solution of which it is necessary to apply not quite traditional methods:

$$20\cdot\left(\frac{x-2}{x+1}\right)^2-5\cdot\left(\frac{x+2}{x-1}\right)^2+48\cdot\frac{x^2-4}{x^2-1}=0.$$

Solution:

R.O.V.V.: $(-\infty;\ -1)\cup(-1;\ 1)\cup(1;\ +\infty)$,

$$20\cdot\left(\frac{x-2}{x+1}\right)^2-5\cdot\left(\frac{x+2}{x-1}\right)^2+48\cdot\frac{(x-2)\cdot(x+2)}{(x+1)\cdot(x-1)}=0$$

$$20\cdot\left(\frac{x-2}{x+1}\right)^2+48\cdot\frac{x-2}{x+1}\cdot\frac{x+2}{x-1}-5\cdot\left(\frac{x+2}{x-1}\right)^2=0.$$

A quadratic equation was formed with respect to $\frac{x-2}{x+1}$. We denote $\frac{x-2}{x+1}=y$, then we will have such a quadratic equation:

$$20y^2+48\cdot\frac{x+2}{x-1}\cdot y-5\cdot\left(\frac{x+2}{x-1}\right)^2=0;$$

$$D=\left(48\cdot\frac{x+2}{x-1}\right)^2-4\cdot20\cdot(-5)\cdot\left(\frac{x+2}{x-1}\right)^2=2304\cdot\left(\frac{x+2}{x-1}\right)^2+400\cdot\left(\frac{x+2}{x-1}\right)^2=2704\cdot\left(\frac{x+2}{x-1}\right)^2=52^2\cdot\left(\frac{x+2}{x-1}\right)^2=\left(52\cdot\frac{x+2}{x-1}\right)^2;$$

$$Y_1=\frac{-48\cdot\frac{x+2}{x-1}-52\cdot\frac{x+2}{x-1}}{2\cdot20}=\frac{-24\cdot\frac{x+2}{x-1}-26\cdot\frac{x+2}{x-1}}{20};\qquad Y_2=\frac{-24\cdot\frac{x+2}{x-1}+26\cdot\frac{x+2}{x-1}}{20}.$$

Back to the substitution:

$$\left[\begin{aligned}&\frac{x-2}{x+1}=\frac{-24\cdot\frac{x+2}{x-1}-26\cdot\frac{x+2}{x-1}}{20};\\&\frac{x-2}{x+1}=\frac{-24\cdot\frac{x+2}{x-1}+26\cdot\frac{x+2}{x-1}}{20}\end{aligned}\right.\qquad\left[\begin{aligned}&\frac{x-2}{x+1}=\frac{-50}{20}\cdot\frac{x+2}{x-1};\\&\frac{x-2}{x+1}=\frac{2}{20}\cdot\frac{x+2}{x-1}\end{aligned}\right.\qquad\left[\begin{aligned}&\frac{x-2}{x+1}=\frac{-5}{2}\cdot\frac{x+2}{x-1};\\&\frac{x-2}{x+1}=\frac{1}{10}\cdot\frac{x+2}{x-1}.\end{aligned}\right.$$

Using the main property of proportion:

$$\left[\begin{aligned}&2\cdot(x-2)\cdot(x-1)=-5\cdot(x+1)\cdot(x+2);\\&10\cdot(x-2)\cdot(x-1)=(x+1)\cdot(x+2)\end{aligned}\right.\qquad\left[\begin{aligned}&2\cdot\left(x^2-x-2x+2\right)=-5\cdot\left(x^2+2x+x+2\right);\\&10\cdot(x-2)\cdot(x-1)=x^2+2x+x+2.\end{aligned}\right.$$

$$\left[\begin{aligned}&2x^2-6x+4=-5x^2-15x-10;\\&10x^2-30x+20=x^2+3x+2\end{aligned}\right.\qquad\left[\begin{aligned}&7x^2+9x+14=0;\\&9x^2-33x+18=0\,\big|:3.\end{aligned}\right.$$

$$\left[\begin{aligned}&7x^2+9x+14=0;\\&3x^2-11x+6=0\end{aligned}\right.\qquad\left[\begin{aligned}&D=81-4\cdot7\cdot14<0,\ x\in\varnothing;\\&D=121-4\cdot3\cdot6=49,\ x_1=\frac{11-7}{6}=\frac{2}{3};\quad x_2=\frac{11+7}{6}=3.\end{aligned}\right.$$

Answer: $\frac{2}{3}$; 3.

Of no less interest is the solution of such an equation:

$$x^3+x^{-3}+x^2+x+x^{-2}+x^{-1}=0.$$

Solution:

Getting rid of negative exponents: $x^3+\frac{1}{x^3}+x^2+x+\frac{1}{x^2}+\frac{1}{x}=0;$

R.O.V.V.: $(-\infty;\ 0)\cup(0;\ +\infty)$. grouping the terms of the equation:

$$\left(x^3+\frac{1}{x^3}\right)+\left(x^2+\frac{1}{x^2}\right)+\left(x+\frac{1}{x}\right)=0;$$

Replacement $x+\frac{1}{x}=y.$ $\left(x+\frac{1}{x}\right)^2=y^2;$

$x^2+2\cdot x\cdot\frac{1}{x}+\frac{1}{x^2}=y^2;$ $x^2+\frac{1}{x^2}=y^2-2.$ $\left(x+\frac{1}{x}\right)^3=x^3+3\cdot x^2\cdot\frac{1}{x}+3\cdot x\cdot\left(\frac{1}{x}\right)^2+\frac{1}{x^3}=y^3;$

$x^3+\frac{1}{x^3}=y^3-3x-3\cdot\frac{1}{x}=y^3-3\cdot\left(x+\frac{1}{x}\right)=y^3-3y.$

Formed a third degree equation:

$y^3-3y+y^2-2+y=0;$ $\left(y^3+y^2\right)-\left(2y+2\right)=0;$ $y^2\cdot\left(y+1\right)-2\cdot\left(y+1\right)=0;$

$\left(y+1\right)\cdot\left(y^2-2\right)=0;$ This equation is equivalent to such a set of equations:

$$\left[\begin{array}{l} y+1=0, \\ y^2-2=0 \end{array}\right. \quad \left[\begin{array}{l} y=-1, \\ \left(y-\sqrt{2}\right)\cdot\left(y+\sqrt{2}\right)=0 \end{array}\right. \quad \left[\begin{array}{l} y=-1, \\ y=-\sqrt{2}, \\ y=\sqrt{2} \end{array}\right. \quad \left[\begin{array}{l} y=-1, \\ y=-\sqrt{2}, \\ y=\sqrt{2} \end{array}\right. \quad \left[\begin{array}{l} x+\frac{1}{x}=-1\Big|\cdot x, \\ x+\frac{1}{x}=-\sqrt{2}\Big|\cdot x, \\ x+\frac{1}{x}=\sqrt{2}\Big|\cdot x. \end{array}\right.$$

$$\left[\begin{array}{l} x^2+x+1=0, \quad D=1-4<0, \ x\in\varnothing; \\ x^2+\sqrt{2x}+1=0, \quad D=2-4<0, \ x\in\varnothing; \\ x^2-\sqrt{2x}+1=0, \quad D=2-4<0, \ x\in\varnothing. \end{array}\right. \qquad \text{Answer: } \varnothing.$$

The following exercise is solved somewhat specifically.

At what values of A, B and C is the correct equality:

$$\frac{x^2+5}{x^3-3x+2}=\frac{A}{x+2}+\frac{B}{(x-1)^2}+\frac{C}{x-1}?$$

Solution:

Reduce the right-hand side of the equality to a common denominator and transform it:

$$\frac{A}{x+2}+\frac{B}{(x-1)^2}+\frac{C}{x-1}=\frac{A\cdot(x-1)^2+B\cdot(x+2)+C\cdot(x+2)\cdot(x-1)}{(x+2)\cdot(x-1)^2}=$$

$$=\frac{A\cdot\left(x^2-2x+1\right)+B\cdot(x+1)+C\cdot\left(x^2+2x-x-2\right)}{(x+2)\cdot(x-1)^2}=\frac{Ax^2-2Ax+A+Bx+2B+Cx^2+Cx-2C}{(x+2)\cdot(x-1)^2}=$$

$$=\frac{(A+C)x^2+(B-2A+C)x+A+2B-2C}{(x+2)\cdot(x-1)^2}=\frac{(A+C)x^2+(B-2A+C)x+A+2B-2C}{(x+2)\cdot\left(x^2-2x+1\right)}=$$

$$=\frac{(A+C)x^2+(B-2A+C)x+A+2B-2C}{x^3+2x^2+x+2x^2-4x+2}=\frac{(A+C)x^2+(B-2A+C)x+A+2B-2C}{x^3-3x+2}.$$

Поменяем местами левую и правую часть этого равенства:

$$\frac{(A+C)x^2+(B-2A+C)x+A+2B-2C}{x^3-3x+2}=\frac{x^2+5}{x^3-3x+2}.$$

Since these fractions are equal, their denominators are equal, then the numerators are also equal. The coefficients for equal degrees of variables are also equal, that is,:

$$\begin{cases} A+C=1, \\ B-2A+C=0, \\ A+2B-2C=5. \end{cases} \quad \begin{cases} A=1-C, \\ B-2\cdot(1-C)+C=0, \\ 1-C+2B-2C=5. \end{cases} \quad \begin{cases} A=1-C, \\ B-2+2C+C=0, \\ 1-3C+2B=5. \end{cases} \quad \begin{cases} A=1-C, \ (1) \\ B+3C-2=0, \ (2) \\ 2B-3C=5. \ (3) \end{cases}$$

Add equations (2) and (3): $\begin{cases} B+3C=2 \\ 2B-3C=5-1 \end{cases}$ $\quad 2+3C=2; \quad 3C=0, \ \ C=0.$

$$3B=6, \quad B=\frac{6}{3}=2. \qquad A=1-0=1, \quad A=1.$$

Answer: A=1, B=2, C=0.

A clever approach to solving equations like:

$\frac{8}{x+4}-\frac{6}{x+3}-\frac{4}{x+2}=\frac{8}{x-4}-\frac{6}{x-3}-\frac{4}{x-2}$ We consider the following method:

Solution:

Let us find the range of admissible values of the equation:

$\left[\begin{array}{l} x+4\neq 0, \\ x+3\neq 0, \\ x+2\neq 0, \\ x-4\neq 0, \\ x-3\neq 0, \\ x-2\neq 0. \end{array}\right. \quad \left[\begin{array}{l} x\neq -4, \\ x\neq -3, \\ x\neq -2, \\ x\neq 4, \\ x\neq 3, \\ x\neq 2. \end{array}\right.$ We collect all the terms of the equation on the left side.

$$\frac{8}{x+4}-\frac{6}{x+3}-\frac{4}{x+2}-\frac{8}{x-4}-\frac{6}{x-3}-\frac{4}{x-2}+\frac{6}{x-3}+\frac{4}{x-2}=0.$$

Let us group by two those terms of the equation whose denominators are the conjugate expressions:

$$\left(\frac{8}{x+4}-\frac{8}{x-4}\right)+\left(\frac{6}{x+3}-\frac{6}{x-3}\right)+\left(\frac{4}{x+2}-\frac{4}{x-2}\right)=0.$$

Each pair of expressions in brackets is reduced to a common denominator:

$$\frac{8x-32-8x-32}{x^2-16}-\frac{6x-18-6x-18}{x^2-9}-\frac{4x-8-4x-8}{x^2-4}=0;$$

$$-\frac{64}{x^2-16}+\frac{36}{x^2-9}+\frac{16}{x^2-4}=0 \Big| :(-4); \qquad \frac{16}{x^2-16}-\frac{9}{x^2-9}-\frac{4}{x^2-4}=0;$$

Let's introduce a new variable: $x^2=y.$ Then $\frac{16}{y-16}-\frac{9}{y-9}-\frac{4}{y-4}=0;$

We use the condition of equality of the fraction to zero:

$$\begin{cases} 16\cdot(y-9)\cdot(y-4)-9\cdot(y-16)\cdot(y-4)-4\cdot(y-16)\cdot(y-9)=0, \\ (y-16)\cdot(y-9)\cdot(y-4)\neq 0. \end{cases}$$

$$16\cdot\left(y^2-4y-9y+36\right)-9\cdot\left(y^2-4y-16y+64\right)-4\cdot\left(y^2-9y-16y+144\right)=0;$$

$$16y^2-208y+576y-9y^2+180y-576-4y^2+100y-576y=0;$$

$$y^2+24y-192=0; \quad D=576+768=1344=4\cdot 336=16\cdot 84=64\cdot 21;$$

$$y_1=\frac{-2-8\sqrt{21}}{2}=-12-4\sqrt{21}<0, \quad y_2=-12+4\sqrt{21}>0,$$

Back to the substitution: $x^2=-12-4\sqrt{21} \quad x\in\varnothing; \qquad x^2=-12+4\sqrt{21};$

$$x_1=\sqrt{4\cdot\sqrt{21}-12}=2\sqrt{\sqrt{21}-3}; \qquad x_2=-2\sqrt{\sqrt{21}-3};$$

Answer: $x_1=\sqrt{4\cdot\sqrt{21}-12}=2\sqrt{\sqrt{21}-3}$; $\quad x_2=-2\sqrt{\sqrt{21}-3}$.

A little bit not so begins the solutions are very similar to the previous equation.

$$\frac{4x-17}{x-4}+\frac{10x-13}{2x-3}-\frac{8x-30}{2x-7}-\frac{5x-4}{x-1}=0.$$

Solution:

R.O.V.V.: $\begin{cases} x\neq 4, \\ x\neq 1{,}5, \\ x\neq 3{,}5, \\ x\neq 1. \end{cases}$ Let's perform such transformations:

$$\frac{4x-16-1}{x-4}+\frac{10x-15+2}{2x-3}-\frac{8x-28-2}{2x-7}-\frac{5x-5+1}{x-1}=0,$$
$$\frac{(4x-16)-1}{x-4}+\frac{(10x-15)+2}{2x-3}-\frac{(8x-28)-2}{2x-7}-\frac{(5x-5)+1}{x-1}=0,$$
$$\frac{4\cdot(x-4)}{x-4}-\frac{1}{x-4}+\frac{5\cdot(2x-3)}{2x-3}+\frac{2}{2x-3}-\frac{4\cdot(2x-7)}{2x-7}+\frac{2}{2x-7}-\frac{5\cdot(x-1)}{x-1}-\frac{1}{x-1}=0,$$

After reduction we get: $4-\frac{1}{x-4}+5+\frac{2}{2x-3}-4+\frac{2}{2x-7}-5-\frac{1}{x-1}=0,$

$$\frac{2}{2x-3}-\frac{1}{x-4}=\frac{1}{x-1}-\frac{2}{2x-7}, \qquad \frac{2\cdot(x-4)-1\cdot(2x-3)}{(2x-3)\cdot(x-4)}=\frac{(2x-7)\cdot 1-2\cdot(x-1)}{(2x-7)\cdot(x-1)},$$
$$\frac{2x-8-2x+3}{(2x-3)\cdot(x-4)}=\frac{2x-7-2x+2}{(2x-7)\cdot(x-1)}, \qquad \frac{-5}{(2x-3)\cdot(x-4)}=\frac{-5}{(2x-7)\cdot(x-1)}\Big|:(-5),$$

$\frac{1}{(2x-3)\cdot(x-4)}=\frac{1}{(2x-7)\cdot(x-1)}$. By the main property of the proportion, we have:

$(2x-7)\cdot(x-1)=(2x-3)\cdot(x-4)$, $\quad 2x^2-2x-7x+7=2x^2-8x-3x+12$,

$-9x-11x=12-7$, $\quad -2x=5$, $\quad x=-2{,}5$. Answer: $-2{,}5$.

Solve the equation: $\frac{24x}{x^3+2x^2-8x}-\frac{15x}{x^3+2x^2-3x}=2.$

Solution:

From the analysis of equations design shows that $x\neq 0$, therefore, the numerator and denominator of each of the fractions can be divided by *x*, that is:

$$\frac{\frac{24x}{x}}{\frac{x^3+2x^2-8x}{x}}-\frac{\frac{15x}{x}}{\frac{x^3+2x^2-3x}{x}}=2, \quad \frac{24}{x^2+2x-8}-\frac{15}{x^2+2x-3}=2.$$

We introduce a new variable:

$x^2+2x-8=y$; $\quad x^2+2x-8+5=y+5$; $\quad x^2+2x-3=y+5$;

Then the equation takes the form: $\frac{24}{y}-\frac{15}{y+5}=2$; $\quad \begin{cases} y\neq 0, \\ y+5\neq 0, \end{cases} \quad \begin{cases} y\neq 0, \\ y\neq -5. \end{cases}$

$$\frac{24\cdot(y+5)-15y}{y\cdot(y+5)}=2.$$

$24\cdot(y+5)-15y=2y\cdot(y+5)$; $\quad 24y+120-15y=2y^2+10y$; $\quad 2y^2+y-120=0$;

$D=1+960=961=31^2$; $\quad y_1=\frac{-1-31}{4}=\frac{-32}{4}=-8$; $\quad y_2=\frac{-1+31}{4}=\frac{30}{4}=\frac{15}{2}$.

Returning to the replacement, we get: $\left[\begin{array}{l} x^2+2x-8=-8, \\ x^2+2x-8=\dfrac{15}{2} \end{array}\right.$ $\left[\begin{array}{l} x^2+2x=0, \\ 2x^2+4x-16=15. \end{array}\right.$

$\left[\begin{array}{l} x\cdot(x+2)=0, \\ 2x^2+4x-31=0 \end{array}\right.$ $\left[\begin{array}{l} x=0, \\ x+2=0, \\ D=16+248=264 \end{array}\right.$ $\left[\begin{array}{l} x=0, \\ x=-2, \\ x=\dfrac{-4-\sqrt{264}}{4}, \\ x=\dfrac{-4+\sqrt{264}}{4} \end{array}\right.$ $\left[\begin{array}{l} x=0, \\ x=-2, \\ x=\dfrac{-2-\sqrt{66}}{2}, \\ x=\dfrac{-2+\sqrt{66}}{2}. \end{array}\right.$

Answer: $x=0, \quad x=-2, \quad x=\dfrac{-2-\sqrt{66}}{2}, \quad x=\dfrac{-2+\sqrt{66}}{2}.$

Solve the equation: $x^2+\dfrac{4\cdot x^2}{(x+2)^2}=5.$

Solution:

Analysis of this equation shows that R.O.V.V. there is:
$(-\infty;\ -2)\cup(-2;\ 0)\cup(0;\ +\infty)$.

On the left side of the equation, select the complete square of the binomial:

$$x^2-2x\cdot\frac{2x}{x+2}+\frac{4x^2}{(x+2)^2}+2\cdot x\cdot\frac{2x}{x+2}=5, \qquad \left(x-\frac{2x}{x+2}\right)^2+\frac{4x^2}{x+2}=5,$$

$$\left(\frac{x^2+2x-2x}{x+2}\right)^2+\frac{4x^2}{x+2}=5, \qquad \left(\frac{x^2}{x+2}\right)^2+\frac{4x^2}{x+2}-5=0.$$ Substitution: $\dfrac{x^2}{x+2}=y.$

We have the equation: $y^2+4y-5=0,$ by Vieta's theorem: $y_1=-5, \quad y_2=1.$

Returning to the substitution, we have a set of equations:

$\left[\begin{array}{l} \dfrac{x^2}{x+2}=-5, \\ \dfrac{x^2}{x+2}=1. \end{array}\right.$ $\left[\begin{array}{l} x^2+5x+10=0, \\ x^2-x-2=0, \end{array}\right.$ $\left[\begin{array}{l} D=25-40<0,\ x\in\varnothing; \\ x_1=-1,\ \in \text{О.Д.З.} \\ x_2=2.\ \in \text{О.Д.З.} \end{array}\right.$ Answer: $-1; \quad 2.$

Solve the equation: $x^2+\dfrac{4}{x^2}+6\cdot\left(x+\dfrac{2}{x}\right)=23.$

Solution:

R.O.V.V.: $x\neq 0.$ We single out the square of the binomial:

$$x^2+2x\cdot\frac{2}{x}+\frac{4}{x^2}-2x\cdot\frac{2}{x}+6\cdot\left(x+\frac{2}{x}\right)=23;$$

$$\left(x+\frac{2}{x}\right)^2+6\cdot\left(x+\frac{2}{x}\right)-4-23=0; \qquad \left(x+\frac{2}{x}\right)^2+6\cdot\left(x+\frac{2}{x}\right)-27=0.$$

Substitution: $x+\dfrac{2}{x}=y.$ The resulting quadratic equation: $y^2+6y-27=0,$ we solve by Vieta's theorem. $y_1=-9, \quad y_2=3.$

$$\left[\begin{array}{l} x+\dfrac{2}{x}=-9, \\ x+\dfrac{2}{x}=3. \end{array}\right. \quad \left[\begin{array}{l} x^2+9x+2=0, \quad D=81-8=73. \\ x^2-3x+2=0. \quad \left[\begin{array}{l} x_1=1, \\ x_2=2. \end{array}\right. \end{array}\right. \quad x_3=\frac{-9-\sqrt{73}}{2}; \quad x_4=\frac{-9+\sqrt{73}}{2}.$$

Answer: $x_1=1;\quad x_2=2;\quad x_3=\dfrac{-9-\sqrt{73}}{2};\quad x_4=\dfrac{-9+\sqrt{73}}{2}.$

Solve the equation: $x^4=\dfrac{11x-6}{6x-11}.$

Solution:

R.O.V.V.: $x\neq\dfrac{11}{6}.\quad x^4\cdot(6x-11)=11x-6;\quad 6x^5-11x^4-11x+6=0;$

We will establish by test that -1 the root of this equation.

$$\begin{array}{l|l} 6x^5-11x^4-11x+6 & x+1 \\ \hline -\;6x^5+6x^4 & 6x^4-17x^3+17x^2-17x+6 \\ \quad -\;\begin{array}{l}-17x^4-11x \\ -17x^4-17x^3\end{array} & \\ \qquad -\;\begin{array}{l}17x^3-11x^2 \\ 17x^3+17x^2\end{array} & \\ \qquad\quad -\;\begin{array}{l}-17x^2-11x \\ -17x^2-17x\end{array} & \\ \qquad\qquad -\;\begin{array}{l}6x+6 \\ 6x+6\end{array} & \\ \qquad\qquad\qquad 0. & \end{array}$$

A symmetric equation was formed: $6x^4-17x^3+17x^2-17x+6=0 \,|:x^2;$

$6x^2-17x+17-\dfrac{17}{x}+\dfrac{6}{x^2}=0;\quad \left(6x^2+\dfrac{6}{x^2}\right)-\left(17x+\dfrac{17}{x}\right)+17=0;\quad 6\cdot\left(x^2+\dfrac{1}{x^2}\right)-17\cdot\left(x+\dfrac{1}{x}\right)+17=0.$

Replacement: $x+\dfrac{1}{x}=t,\quad x^2+2x\cdot\dfrac{1}{x}+\dfrac{1}{x^2}=t^2;\quad x^2+\dfrac{1}{x^2}=t^2-2;\quad 6t^2-12-17t+17=0;$

$6t^2-17t+5=0.\quad D=289-120=169=13^2;\quad t_1=\dfrac{17-13}{12}=\dfrac{4}{12}=\dfrac{1}{3};\quad t_2=\dfrac{17+13}{12}=\dfrac{30}{12}=\dfrac{5}{2};$

$$\left[\begin{array}{l} x+\dfrac{1}{x}=\dfrac{1}{3}, \\ x+\dfrac{1}{x}=\dfrac{5}{2}. \end{array}\right. \quad \left[\begin{array}{l} 3x^2+3-x=0, \\ 2x^2+2-5x=0. \end{array}\right. \quad \left[\begin{array}{l} 3x^2-x+3=0, \quad D=1-36<0,\ x\in\varnothing; \\ 2x^2-5x+2=0. \quad D=25-16=6, \quad \left[\begin{array}{l} x_1=\dfrac{5-3}{4}=\dfrac{1}{2}\in О.Д.З. \\ x_2=\dfrac{5+3}{4}=2\in О.Д.З. \end{array}\right. \end{array}\right.$$

Answer: $-1;\ \dfrac{1}{2};\ 2.$

Two-term equations

A whole rational equation of the form $Ax^n + B = 0$, де $A \neq 0$ і $B \neq 0$, It called binomial. To solve the two-term equation, it is necessary to make the following replacement: $x = t \cdot \sqrt[n]{\frac{B}{A}}$.

For example, $3 \cdot x^8 + 16 = 0$, $x = t \cdot \sqrt[8]{\frac{16}{3}} = \frac{t \cdot \sqrt[8]{16}}{\sqrt[8]{3}} = \frac{t \cdot \sqrt[8]{2^4}}{\sqrt[8]{3}} = \frac{t \cdot \sqrt{2}}{\sqrt[8]{3}}$.

Then this equation takes the form:

$$3 \cdot \left(\frac{t\sqrt{2}}{\sqrt[8]{3}}\right)^8 + 16 = 0, \quad \frac{3 \cdot t^8 \cdot 16}{3} + 16 = 0, \quad 16t^8 + 16 = 0, \quad t^8 + 1 = 0.$$

This equation, and hence the original one, has no real roots.
Answer: $\varnothing$.

Trinomial equation

A whole rational equation of the form: $A \cdot x^{2n} + B \cdot x^n + C = 0$, де $A \neq 0$, $B \neq 0$, $C \neq 0$, such an equation is called three-term. By introducing a new variable $x^n = t$ reduced to a square. For example, $x^6 - 35x^3 + 216 = 0$, is solved as follows: replacement $x^3 = t$ leads to the quadratic equation: $t^2 - 35t + 216 = 0$. By Vieta's theorem, we have: $t_1 = 8$, $t_2 = 27$. Returning to the replacement, we obtain a set of equations:

$$\left[\begin{array}{l} x^3 = 8, \\ x^3 = 27. \end{array}\right. \quad \left[\begin{array}{l} x^3 - 8 = 0, \\ x^3 - 27 = 0. \end{array}\right. \quad \left[\begin{array}{l} x^3 - 2^3 = 0, \\ x^3 - 3^3 = 27. \end{array}\right. \quad \left[\begin{array}{l} (x-2)(x^2 + 2x + 4) = 0, \\ (x-3)(x^2 + 3x + 9) = 0. \end{array}\right.$$

$$\left[\begin{array}{l} x - 2 = 0, \\ x^2 + 2x + 4 = 0. \end{array}\right. \quad \left[\begin{array}{l} x - 3 = 0, \\ x^2 + 3x + 9 = 0. \end{array}\right.$$

$\left[\begin{array}{l} x = 2, \\ x = 3. \end{array}\right.$ Equations $x^2 + 2x + 4 = 0$ та $x^2 + 3x + 9 = 0$, have no real roots.

Answer: 2 і 3.

Self-study assignments:

Solve Equations:

5.36. $\frac{2x+1}{3-x} = \frac{4-x}{x+1}$. Answer: -11; 1.

5.37. $\frac{3x}{x-1} - \frac{2x}{x+2} = \frac{3x-6}{(x-1)\cdot(x+2)}$. Answer: -3.

5.38. $\frac{x^2+2x+2}{x+2} - \frac{2x+9}{x+3} = \frac{x^2+x-1}{x+1} - \frac{2x+4}{x+4}$. Answer: 0; $-2{,}5+0{,}5\sqrt{3}$; $-2{,}5-0{,}5\sqrt{3}$.

5.39. $\frac{x-2}{x\cdot(x-1)} + \frac{1}{x^2-1} = \frac{2}{x\cdot(x+1)}$. Answer: 2.

5.40. $\frac{1}{x-3} + \frac{1}{x+3} = \frac{1}{5-x} - \frac{1}{5+x}$. Answer: 0; $-\sqrt{17}$; $\sqrt{17}$.

5.41. $2x-\frac{x^2-3x+1}{x-3}=\frac{x^2-x+1}{x-1}+\frac{1}{4\cdot(x-2)}$. Answer: $1\frac{2}{3}$; $2\frac{1}{3}$.

5.42. $\frac{(x+3)\cdot(x-5)}{9\cdot(x+4)\cdot(x-6)}-\frac{2(x+5)\cdot(x-7)}{13\cdot(x+6)\cdot(x-8)}=\frac{92}{585}-\frac{(x+1)\cdot(x-3)}{5\cdot(x+2)\cdot(x-4)}$. Answer: 0; $-2{,}5+0{,}5\sqrt{3}$.

5.43. $\frac{(x^2-x+1)^2}{(x-1)^2\cdot(x^2+1)}=\frac{9}{5}$. Answer: 2; $\frac{1}{2}$.

5.44. $\frac{(x+1)^5}{x^5+1}=16$. Answer: 1; $\frac{1}{3}$.

5.45. $x^3=\frac{17x-10}{10x-17}$. Answer: 2; $\frac{1}{2}$.

5.46. $\frac{x^2-5x+7}{6x}=\frac{x^2-3x+7}{x^2+2x+7}$. Answer: 1; 7.

5.47. $x^2+\frac{9x^2}{(x-3)^2}=7$. Answer: $-0{,}5-0{,}5\cdot\sqrt{13}$; $-0{,}5+0{,}5\sqrt{13}$.

5.48. $x^3-x^2-2=\frac{8}{x^3-x^2}$. Answer: -1; 2.

5.49. $4x^2+12x+\frac{12}{x}+\frac{4}{x^2}=47$. Answer: $\frac{1}{2}$; 2; $\frac{-11-\sqrt{105}}{4}$; $\frac{-11+\sqrt{105}}{4}$.

5.50. $\frac{3}{x\cdot(x-5)}+\frac{1}{x^2-5x+4}+\frac{4}{(x-2)\cdot(x-3)}=0$. Answer: $\frac{5-\sqrt{7}}{2}$; $\frac{5+\sqrt{7}}{2}$; $\frac{5-\sqrt{17}}{2}$; $\frac{5+\sqrt{17}}{2}$.

5.51. $\frac{1}{x\cdot(x+2)}-\frac{1}{(x+1)^2}=\frac{1}{12}$. Answer: -3; 1.

Diophantine equations

Diophantine equations are often encountered among algebraic equations with rational coefficients. Convenient for their recognition is the concept of a system of Diophantine equations: it is such a system of algebraic equations with rational coefficients, which contains more variables than the equations themselves. Following this logic, we can say that a Diophantine equation is an algebraic equation with rational coefficients, in which there are more than one variable. Following this logic, we can say that a Diophantine equation is an algebraic equation with rational coefficients, in which there are more than one variable:

1). Express one variable through another;

2). Select the value of the second variable so that the first variable takes integer values.

Find all integer solutions to an equation:
$15x-7y=13$.

Solution:

$7y=15x-13$; $\quad y=\frac{15x-13}{7}=\frac{15x}{7}-\frac{13}{7}=2x+\frac{x}{7}-2+\frac{1}{7}=2x-2+\frac{x+1}{7}$.

Analysis of this expression shows that y takes integer values when an expression is an integer $\frac{x+1}{7}$, that is, when $x+1=7k$, де $k \in z$. $x=7k-1$, then the equation will have the form: $y=2\cdot(7k-1)-2+\frac{7k}{7}=14k-2-2+k=15k-4$.

Answer: $(7k-1;\ 15k-4;\ k \in z)$.

Find all non-negative integer solutions of the equation: $19x+84y=1984$.

Solution:

Imagine the number 1984 as a sum of two terms, one of which is a multiple of a number 19, and the second – 84. $1984=1900+84$ and over a given Diophantine equation implement changes that allow it to find all integer solutions:

$19x+84y=1900+84,\ 19x-1900+84y=84,\ 19\cdot(x-100)+84y=84,$

$84y=84-19\cdot(x-100),\ y=\frac{84}{84}-\frac{19\cdot(x-100)}{84}=1-19\cdot\frac{x-100}{84}.$

This expression is integer when $x-100$ multiples 84, that is, $x-100=84k$, where $k \in z$, from here:

$x=84k+100$, then $y=1-19\cdot\frac{84k}{84}$, $y=1-19k$. $(100+84k;\ 1-19k)$, $k \in z-$ all integer solutions of the equation. To find non-negative integer solutions of this equation, we solve the following system of inequalities:

$\begin{cases} 84k+100 \ge 0, \\ 1-19k \ge 0. \end{cases}$ $\begin{cases} 84k \ge -100, \\ -19k \ge -1. \end{cases}$ $\begin{cases} k \ge -\frac{100}{84}, \\ k \le \frac{1}{19}. \end{cases}$ Because the $k \in z$, then the solutions of this system: $k \in \{-1;\ 0\}$. Substituting these values k в into the expression of all whole solutions, we get positive integer solutions:

$(100+84\cdot(-1);\ 1-19\cdot(-1))$; $(16;\ 20)$.

$(100+84\cdot 0;\ 1-19\cdot 0)$; $(100;\ 1)$. Answer: $(16;\ 20)$, $(100;\ 1)$.

There are often problems whose solution is based on Diophantine equations:

Find all positive 3-digit numbers, which, when divided by 37, gives a remainder of 2, and when divided by 11, a remainder of 5.

Solution:

Let the $x-$ required number. According to the problem statement, it can be written as: $x=37k+2$ and $x=11m+5$, where $k \in N$ and $m \in N$.

Since the task is about the same number *x*, then you can write the Diophantine equation: $11m+5=37k+2$ express *m* through *k*: $11m=37k+2-5$, $11m=37k-3$,

$m=\frac{37k-3}{11}=3k+\frac{4k}{11}-\frac{3}{11}=3k+\frac{4k-3}{11}$, this expression takes on integer values when $4k-3=11p$, where $p \in z$, then:

$4k=11p+3,\ k=\frac{11p+3}{4}=3p-\frac{p}{4}+1-\frac{1}{4}=3p+1-\left(\frac{p}{4}+\frac{1}{4}\right)=3p+1-\frac{p+1}{4}.$

This expression takes on integer values when $p+1=4q$, where $q \in z$. $p=4q-1$, then $k=\frac{11\cdot(4q-1)+3}{4}=\frac{11\cdot 4q}{4}-\frac{11}{4}+\frac{3}{4}=11q-\frac{8}{4}=11q-2$.

$x=37\cdot(11q-2)+2=407q-74+2=407q-72$. $x=407q-72-$ formula for all positive numbers satisfying the conditions of the problem. From this set of numbers, you need to select only three digits:

at $q=1$ we have $x=407\cdot 1-72=335$;

at $q=2$ we have $x=407\cdot 2-72=742$;

at $q=3$ we have $x=407\cdot 3-72=1149-$ four digit number.

Answer: 335; 742.

The second way to solve Diophantine equations can be called the way of factoring the left side of the equation. It lies in the fact that:

1). Make the right side of the equation an integer;

2). The left side of the equation is factorized;

3). Replace the resulting equation with a set of systems of simple equations.

For example, solve the equation in integers: $xy=x+2y+1991$.

Solution:

Members containing variables move the left side $xy-x-2y=1991$.

Add to both sides of the equation the number 2, and then factor its left side: $(xy-2y)+(-x+2)=1993$; $y\cdot(x-2)-(x-2)=1993$; $(x-2)\cdot(y-1)=1993$ (A). Since the number 1993 – is simple, then it can be represented as a product in only two ways: $1993=1\cdot 1993$ and $1993=-1\cdot(-1993)$. On the set of integers, the Diophantine equation (A) is equivalent to a set of such systems of equations:

$$\left[\begin{array}{l} \begin{cases} x-2=1, \\ y-1=1993. \end{cases} \\ \begin{cases} x-2=1993, \\ y-1=1. \end{cases} \\ \begin{cases} x-2=-1, \\ y-1=-1993. \end{cases} \\ \begin{cases} x-2=-1993, \\ y-1=-1. \end{cases} \end{array}\right. \qquad \left[\begin{array}{l} \begin{cases} x=3, \\ y=1994. \end{cases} \\ \begin{cases} x=1995, \\ y=2. \end{cases} \\ \begin{cases} x=1, \\ y=-1992. \end{cases} \\ \begin{cases} x=-1991, \\ y=0. \end{cases} \end{array}\right.$$

Answer: $(3;\, 1994)$, $(1995;\, 2)$, $(1;\, -1992)$, $(1991;\, 0)$.

Find the goals of solving an equation: $3xy+16x+13y+61=0$.

Solution:

We multiply both sides of the equation by 3: $9xy+48x+39y+183=0$.

Add to both sides of the equation the number 25: $9xy+48x+39y+208=25$;

$(9xy+39y)+(48x+208)=25$; $3y(3x+13)+16\cdot(3x+13)=25$; $(3x+13)\cdot(3y+16)=25$;

This equation is equivalent to a set of such systems of equations:

$$\left[\begin{array}{l}\begin{cases}3x+13=1,\\3y+16=25.\end{cases}\\\begin{cases}3x+13=25,\\3y+16=1.\end{cases}\\\begin{cases}3x+13=-1,\\3y+16=-25.\end{cases}\\\begin{cases}3x+13=-25,\\3y+16=-1.\end{cases}\\\begin{cases}3x+13=5,\\3y+16=5.\end{cases}\\\begin{cases}3x+13=-5,\\3y+16=-5.\end{cases}\end{array}\right. \quad \left[\begin{array}{l}\begin{cases}3x=-12,\\3y=3.\end{cases}\\\begin{cases}3x=12,\\3y=-15.\end{cases}\\\begin{cases}3x=-14,\\3y=-41.\end{cases}\\\begin{cases}3x=-38,\\3y=-17.\end{cases}\\\begin{cases}3x=2,\\3y=-11.\end{cases}\\\begin{cases}3x=-18,\\3y=-21.\end{cases}\end{array}\right. \quad \left[\begin{array}{l}\begin{cases}x=-4,\\y=3.\end{cases}\\\begin{cases}x=4,\\y=-5.\end{cases}\\\begin{cases}x=-\frac{14}{3}\notin z,\\y=-\frac{41}{3}\notin z.\end{cases}\\\begin{cases}x=-\frac{38}{3}\notin z,\\y=-\frac{17}{3}\notin z.\end{cases}\\\begin{cases}x=\frac{2}{3}\notin z,\\y=-\frac{11}{3}\notin z.\end{cases}\\\begin{cases}x=-6,\\y=-7.\end{cases}\end{array}\right.$$

Answer: $(-4;\ 3),\ (4;\ -5),\ (-6;\ -7)$.

The third way to solve Diophantine equations is called localization. Its essence lies in the fact that using the features of the equation, they localize the set on which its values can be contained and then find them by direct verification.

Solve an equation in whole numbers: $3x^2+5y^2=345$.

Solution:

Expression $3x^2 \geq 0$ for all values x. Then $5y^2 \leq 345$. Dividing both sides of the inequality by 5, we get $y^2 \leq 69. 0 \leq y \leq \sqrt{69}, 0 \leq y \leq 8$. Because the $3x^2$ is divisible by 3 and 345 is divisible by 3, then $5y^2$ is divisible by 3, that is y^2 is divisible by 3 and therefore divisible by 3. In between $[0;\ 8]$ y=$\{0;\ 3;\ 6\}$.

Substituting in equation instead of the original values of y: 0; 3; 6. $y=0$:

$3x^2+5\cdot 0=345;$

$3x^2=345;\quad x^2=345:3;\quad x^2=115;\quad x=\sqrt{115}\notin z;\quad y=3:\quad 3x^2+5\cdot 3^2=345;$

$3x^2=345-45;\quad x^2=300:3;\quad x^2=100;\quad x_1=10,\quad x_2=-10.$

Let's get two solutions $(10;\ 3), (-10;\ 3)$.

Combining signs in these solutions there will be two more solutions $(10;\ -3), (-10;\ -3)$.

Answer: $(10;\ 3), (-10;\ 3), (10;\ -3), (-10;\ -3)$.

Solve the equation: $x+y=x^2-xy+y^2$

Solution:

Let us perform identical transformations over this equation: $x^2-xy+y^2-x-y=0$;

Let us group some terms of the equation: $\left(x^2-xy-x+y^2-y\right)=0$;

$x^2-(y+1)x+y^2-y=0.$

A quadratic equation is formed with respect to x.
It has real roots when its discriminant is inherent, that is
$D=(y+1)^2-4\cdot 1\cdot(y^2-y)\geq 0;\quad y^2+2y+1-4y^2+4y\geq 0;$
$-3y^2+6y+1\geq 0;\ 3y^2-6y-1\leq 0;\ D=36-4\cdot 3\cdot(-1)=48;$
$$y_1=\frac{6-\sqrt{48}}{6}=\frac{6-4\sqrt{3}}{6}=\frac{3-2\sqrt{3}}{3};\quad y_2=\frac{6+\sqrt{48}}{6}=\frac{6+4\sqrt{3}}{6}=\frac{3+2\sqrt{3}}{3};$$
Interval of localization: $\left[\frac{3-2\sqrt{3}}{3};\ \frac{3+2\sqrt{3}}{3}\right].$

Entire solutions of inequality: 0,1; 2.
Substituting the found values of $\boldsymbol{y}$ into the original equation, we find the corresponding values of $\boldsymbol{x}$:

$y=0:\quad x^2-(1+0)\cdot x+0=0;\quad x^2-x=0;\quad x_1=0;\quad x_2=1.$ (0; 0), (1; 0).

$y=1:\quad x^2-2x=0;\quad x_3=0;\quad x_4=2.$ (0; 1), (2; 1).

$y=2:\quad x^2-3x+4-2=0;\quad x^2-3x+2=0;\quad x_5=1;\quad x_6=2.$ (1; 2), (2; 2).

Answer: (0; 0), (1; 0), (0; 1), (2; 1), (1; 2), (2; 2).

Solve the equation: $\frac{x+y}{x^2-xy+y^2}=\frac{3}{7};$

Solution:

R.O.V.V.: $x\neq 0;\ \ y\neq 0.$ By the main property of the progression, we have:
$3\cdot(x^2-xy+y^2)=7\cdot(x+y);\ \ 3x^2-3xy+3y^2-7x-7y=0;\ \ 3x^2-(3xy+7x)+(3y^2-7y)=0;$
$3x^2-(3y+7)\cdot x+3y^2-7y=0;$
This equation has real roots when: $D\geq 0.$
$D=(3y+7)^2-4\cdot 3\cdot(3y^2-7y)\geq 0;\quad 9y^2+42y+49-36y^2+84y\geq 0;$
$9y^2-27y^2+126y+49\geq 0\,|\cdot(-1);\quad 27y^2-126y-49\leq 0.$
$D_1=15876-4\cdot 27\cdot(-49)=15876+5292=21168=36\cdot 588.$
$$y_1=\frac{126-6\cdot\sqrt{588}}{54}=\frac{6\cdot\left(21-\sqrt{588}\right)}{54}=\frac{21-\sqrt{588}}{9};\quad y_2=\frac{21+\sqrt{588}}{9}.$$
Interval of localization: $\left[\frac{21-\sqrt{588}}{9};\ \frac{21+\sqrt{588}}{9}\right].$

Integers from this interval $y=\{1;\ 2;\ 3;\ 4;\ 5\}.$

Substituting them into the original equation: $y=1:\quad \frac{x+1}{x^2-x+1}=\frac{3}{7},$
$3x^2-3x+3-7x-7=0,\ \ 3x^2-10x-4=0.\ \ D=100+48=148,\ \ x_1\notin z;\ \ x_2\notin z.$
$y=2:\quad \frac{x+2}{x^2-2x+4}=\frac{3}{7};\ \ 3x^2-6x+12=7x+14;\ \ 3x^2-13x-2=0.$
$D=169+24=193;\ \ x_3\notin z;\ \ x_4\notin z.\ \ \ y=3:\quad \frac{x+3}{x^2-3x+9}=\frac{3}{7};\ \ 3x^2-9x+27=7x+21;$
$3x^2-16x+6=0.\ \ D=256-72=184;\ \ x_5\notin z;\ \ x_6\notin z.\ \ \ y=4:\quad \frac{x+4}{x^2-4x+16}=\frac{3}{7};$
$3x^2-12x+48=7x+28;\quad 3x^2-19+20=0.$

$D = 361 - 240 = 121;\quad x_1 = \dfrac{19-11}{6} = \dfrac{8}{6} = \dfrac{4}{3} \notin z;\quad x_2 = \dfrac{19+11}{9} = 5 \notin z;\quad (5; 4).$

$y = 5:\quad \dfrac{x+5}{x^2 - 5x + 25} = \dfrac{3}{7};\quad 3x^2 - 15x + 75 = 7x + 35;\quad 3x^2 - 22x + 40 = 0.$

$D = 484 - 480 = 4;\quad x_1 = \dfrac{22-2}{6} = \dfrac{10}{3} \notin z.\quad x_2 = \dfrac{22+2}{6} = \dfrac{24}{6} = 4 \in z.$

Diophantine equation has only two integer solutions: $(5; 4),\ (4; 5).$
Answer: $(5; 4),\ (4; 5).$

We have considered only three ways to solve Diophantine equations. In fact, there are much more of them. Systems of Diophantine equations can be solved in the above three ways.

Solve a system of equations in integers: $\begin{cases} x^2 - y^2 - z^2 = 1, \\ z + y - x = 3. \end{cases}$

Solution:

The left side of the first equation of the system can be transformed as follows:

$x^2 - y^2 - z^2 = x^2 - \left(y^2 + z^2\right) = x^2 - \left(y^2 + z^2 - 2yz + 2yz\right) = x^2 - \left((y+z)^2 - 2yz\right) =$
$= x^2 - (y+z)^2 + 2yz = (x - y - z)\cdot(x + y + z) + 2yz.$

Identical transformation of the second equation of the system $z + y - x = 3,$ дает $x = z + y - 3.$

Taking into account these factors, the original system of equations can be replaced by a system equivalent to it, that is, $\begin{cases} (z + y - 3 - y - z)\cdot(z + y - 3 + y + z) + 2yz = 1, \\ x = y + z - 3. \end{cases}$

$-6x - 6y + 9 + 2yz = 1;\quad 3z + 3y - 4 - yz = 0;\quad 3y + 3z - yz - 9 = -5 \,|\cdot(-1);$
$-3y - 3z + yz + 9 = 5;\quad (yz - 3y) + (-3z + 9) = 5;\quad y(z-3) - 3(z-3) = 5;\quad (z-3)\cdot(y-3) = 5.$

A system of Diophantine equations was formed:

$\begin{cases} (y-3)(z-3) = 5, \\ x = y + z - 3. \end{cases}$ (A).

The first equation of this system is equivalent to such a set of systems:

$$\left[\begin{array}{l} \begin{cases} y - 3 = 1, \\ z - 3 = 5. \end{cases} \\ \begin{cases} y - 3 = 5, \\ z - 3 = 1. \end{cases} \\ \begin{cases} y - 3 = -1, \\ z - 3 = -5. \end{cases} \\ \begin{cases} y - 3 = -5, \\ z - 3 = -1. \end{cases} \end{array}\right. \qquad \left[\begin{array}{l} \begin{cases} y = 4, \\ z = 8. \end{cases} \\ \begin{cases} y = 8, \\ z = 4. \end{cases} \\ \begin{cases} y = 2, \\ z = -2. \end{cases} \\ \begin{cases} y = -2, \\ z = 2. \end{cases} \end{array}\right.$$

Substituting the found values of $\boldsymbol{y}$ and $\boldsymbol{z}$ into the second equation of system (A), we obtain the corresponding values of $\boldsymbol{x}$: $x=4+8-3=9$ $(9; 4; 8),$
$x=8+4-3=9$ $(9; 8; 4),$
$x=2-2-3=-3$ $(-3; 2; -2),$
$x=-2+2-3=-3$ $(-3; -2; 2).$

Answer: $(-3; -2; 2)$, $(-3; 2; -2)$, $(9; 8; 4)$, $(9; 4; 8)$.
Many problems are solved using Diophantine equations much easier than traditional methods..
The sum of the house numbers of one block on one side of the street is 235. Specify house numbers for this quarter.

Solution:

Since the number 235 is odd, the house numbers of this quarter are expressed odd numbers. Let the number of the first house be expressed by the number $2x+1$, and the last $-\ 2y-1$. calculate the sum of the first n odd natural numbers $1+3+5+\ldots+(2n-1)$. This series is an arithmetic progression with the first term $a_1 = 1$, difference $d = 2$ and the number of members n. Its sum $S_n = \frac{2a_1 + d\cdot(n-1)}{2}\cdot n$,

$$S_n = \frac{2\cdot 1 + 2\cdot n - 1}{2}\cdot n = \frac{2+2n-2}{2}\cdot n = n^2 \ (A).$$

If on the one hand nahodytsya's houses, it s According Amount (A)$= x^2$.
If there are y houses, then their sum y^2, and the difference $y^2 - x^2 = 235 -$ Diophantine equation. $2x+1$ та $2y-1 -$ " generators " odd numbers, functions from variables x and y. Equation $(y-x)\cdot(y+x) = 235 -$ is equivalent to such a set of systems of equations:

$$\left[\begin{array}{llll}
\begin{cases} y-x=1, \\ y+x=235. \end{cases} & \begin{cases} 2y=236, \\ y+x=235. \end{cases} & \begin{cases} y=118, \\ x=235-118. \end{cases} & \begin{cases} y=118, \\ x=117. \end{cases} \\
\begin{cases} y-x=235, \\ y+x=1. \end{cases} & \begin{cases} 2y=236, \\ y+x=1. \end{cases} & \begin{cases} y=118, \\ x=1-118=-117 \notin N \end{cases} & \\
\begin{cases} y-x=5, \\ y+x=47. \end{cases} & \begin{cases} 2y=52, \\ x+y=47. \end{cases} & \begin{cases} y=26, \\ x=47-26. \end{cases} & \begin{cases} y=26, \\ x=21. \end{cases} \\
\begin{cases} y-x=47, \\ y+x=5. \end{cases} & \begin{cases} 2y=52, \\ x+y=5. \end{cases} & \begin{cases} y=26, \\ x=5-26=-21 \notin N. \end{cases} &
\end{array}\right.$$

Find the value of the argument, you can find the corresponding values of the function $2x+1 = 2\cdot 21 + 1 = 43$. Since the house numbers are an arithmetic progression with a difference of 2, their next four numbers are: 43+2=45, 45+2=47, 47+2=49, 49+2=51.
Answer: can be one house with a room 235, or five houses with rooms: 43, 45, 47, 49, 51.
Exercise: There are nuts in two boxes. If you transfer 100 nuts from the first to the second, then the second box will contain twice as many nuts as the first. If, on the contrary, transfer several nuts from the other box to the first, then the first will have 6 times more than the second.
What is the smallest number of nuts possible in the first box?
How many nuts in this case are in the second box?

Solution:

Let the x – number of nuts in the first box, y – number of nuts in the second box, $z -$ number of nuts transferred from the second box to the first, then $(x-100)$ nuts

became in the first box after they took from it 100 nuts. $(y+100)$ nuts became in the second box after 100 nuts were transferred into it.
According to the condition of the problem, we compose the equation
$2\cdot(x-100)=y+100$ (1).
$(y-z)$ nuts became in the second box when they took z nuts.
$(x+z)$ nuts became in the first box when they put it in z nuts.
According to the condition of the problem, we compose the equation
$6\cdot(y-z)=x+z$ (2).
The values of the variables must satisfy equations (1) and (2), and therefore they must satisfy such a system of Diophantine equations: $\begin{cases}2\cdot(x-100)=y+100,\\6\cdot(y-z)=x+z.\end{cases}$

$\begin{cases}2x-200=y+100,\\6y-6z=x+z\end{cases}$ $\begin{cases}y=2x-300,\\6\cdot(2x-300)-6z=x+z.\end{cases}$ $\begin{cases}y=2x-300,\\12x-1800-6z=x+z.\end{cases}$

$\begin{cases}y=2x-300,\\11x-7z=1800.\end{cases}$ $\begin{cases}y=2x-300,\\11x=1800+7z.\end{cases}$ $\begin{cases}y=2x-300,\\x=\frac{1800}{11}+\frac{7z}{11}.\end{cases}$

$\begin{cases}y=2x-300,\\x=163+\frac{7}{11}+\frac{7z}{11}.\end{cases}$ $\begin{cases}y=2x-300,\\x=163+\frac{7\cdot(z+1)}{11}.\end{cases}$ $\begin{cases}y=2x-300,\\x=163+7\cdot\frac{z+1}{11}.\end{cases}$

From the second equation of the system it follows that the smallest integer value of x is the number that is formed when $\frac{z+1}{11}=1$, that is, when $z+1=11$, $z=11-1$, $z=10$.
Wherein $x=163+7\cdot1=170$. $170-$ smallest integer number of nuts in the first basket.
Number of nuts in the second basket $y=2\cdot170-300=340-300=40$.
Answer: 170 nuts, 40 nuts.

Self-study assignments:

Find all integer solutions to an equation:
5.52. $1999x+2000y=2001$. Answer: $(2000k-1;\ 2-1999k)$, $k\in z$.
5.53. $xy+3x-y=20$. Answer: $(-16;\ (-4))$; $(0;\ (-20))$; $(2;\ 14)$; $(18;\ -2)$.
5.54. $6x^2+5y^2=74$. Answer: $(3;\ 2)$; $(3;\ (-2))$; $(-3;\ 2)$; $(-3;\ (-2))$.
Find all non-negative integer solutions of equations:
5.55. $19x+99y=1999$. Answer: $(100;\ 1)$.
5.56. $x\cdot xy=x+y$. Answer: $(0;\ 0)$; $(2;\ 2)$.
5.57. $19x+93y=4xy$. Answer: $(0;\ 0)$; $(28;\ 28)$; $(24;\ 152)$; $(31;\ 19)$; $(465;\ 5)$.
Solve the system of equations in natural numbers:
5.58. $\begin{cases}x^3-y^3-z^3=3xyz,\\x^2=2(y+z).\end{cases}$ Answer: $(2;\ 1;\ 1)$.
5.59. $\begin{cases}x+y+z=3,\\x^3+y^3+z^3=3.\end{cases}$ Answer: $(1;\ 1;\ 1)$, $(-5;\ 4;\ 4)$, $(4;\ -5;\ 4)$, $(4;\ 4;\ -5)$.

5.60. $\begin{cases} xy + z = 94, \\ x + yz = 95. \end{cases}$ Answer: $(95;\ 0;\ 94)$.

Solve tasks:

5.61. Find a two-digit number that is twice the product of its digits.

Answer: 36.

5.62. What is the largest number of chess sets for 5 and 8 hryvnia can be purchased for 103 hryvnias.?

Answer: 1 set for 8 UAH. and 19 sets of 5 UAH each, 20 sets in total.

5.63. The mass of 100 weights, which are in one pile, is equal to 500 grams. It is known that there are only 1 gram, 10 gram and 50 gram weights.

How many weights of each mass are combined?

Answer: 60, 39 and 1.

5.64. Petka is 3 years older than Kolka, and 6 years for Vaska. The product of the ages of Tarasik and Kolka is 9 more than the product of the ages of Petka and Vaska.

How old is Petka older than Tarasik?

Answer: older by 3 years.