Section 8
Logarithmic Equations

Equations of the form $\log_a x = b,$ де $a > 0$ и $a \neq 1$ called the simplest logarithmic. In general, we can say that a transcendental equation, at least one member of which contains a variable under the sign of the logarithm, is called logarithmic.

One of the ways to solve logarithmic equations is to use the basic logarithmic identity $a^{\log_a n = N}$.

For example: $4^{\log_{64}(x-3)} + \log_2 5 = 50.$

Solution:

Range of valid values of the equation (R.O.V.V.): $x - 3 > 0,\ \ x > 3,\ \ x \in (3;\ +\infty).$

We transform the left side of the equation:

$$4^{\log_{64}(x-3)} +^{\log_2 5} = 4^{\log_2 5} = 4^{\log_{64}(x-3)} \cdot 4^{\log_2 5} = \left(\sqrt[3]{64}\right)^{\log_{64}(x-3)} \cdot \left(2^2\right)^{\log_2 5} =$$

$$\left(64^{\frac{1}{3}}\right)^{\log_{64}(x-3)} \cdot 2^{2\log_2 5} = 64^{\frac{1}{3}\log_{64}(x-3)} \cdot 2^{\log_2 5^2} = 64^{\log_{64}(x-3)^{\frac{1}{3}}} \cdot 2^{\log_2 25} = (x-3)^{\frac{1}{3}} \cdot 25.$$

This equation has the form: $(x-3)^{\frac{1}{3}} \cdot 25 = 50,\ \ (x-3)^{\frac{1}{3}} = \frac{50}{25},\ \ (x-3)^{\frac{1}{3}} = 2,$

$\left((x-3)^{\frac{1}{3}}\right)^3 = 2^3;\ \ x - 3 = 8,\ \ x = 11.$ $11 \in$ R.O.V.V.

But it doesn't hurt to do a verification.

If $x = 11,$ then $4^{\log_{64}(11-3)} +^{\log_2 5} = 4^{\log_{64} 8} +^{\log_2 5} = 4^{\frac{1}{2}} \cdot 4^{\log_2 5} = 2 \cdot 2^{\log_2 25} = 2 \cdot 25 = 50.$ $50 = 50.$ $x = 11 -$ root of the equation.

Answer: 11.

In the process of solving logarithmic equations, the appearance of extraneous roots is possible, and therefore verification is advisable in any case.

A good lever is the formula: $\log_a N = \dfrac{\log_b N}{\log_b A}$ transition from one base of the logarithm to another, greatly simplifies the solution of trigonometric equations.

For example, $2\log_x 25 - 3\log_{25} x = 1.$

Solution:

R.O.V.V.: $x \in (0;\ 1) \cup (1;\ +\infty).$ $2 \cdot \dfrac{\log_{25} 25}{\log_{25} x} - 3 \cdot \log_{25} x = 1;$ $\dfrac{2 \cdot 1}{\log_{25} x - 3 \cdot \log_{25} x} = 1;$

We denote: $\log_{25} x = y,$ then $\dfrac{2}{y} - 3y - 1 = 0;$ $\begin{cases} 3y^2 - y + 2 = 0; \\ y \neq 0; \end{cases}$ $D = 1 + 24 = 25;$

$y_1 = \dfrac{1-5}{-6} = \dfrac{2}{3} \in$ R.O.V.V. $y_2 = \dfrac{1+5}{-6} = -1 \notin$ R.O.V.V. $\log_{25} x = \dfrac{2}{3};\ \ x = 25^{\frac{2}{3}}.$

Answer: $x = 25^{\frac{2}{3}}.$

Let's solve the following equation: $x^{1-\frac{1}{3}\lg x^2} - \frac{1}{\sqrt[3]{100}} = 0.$

Solution:

Let's square both sides of the equation: $\left(x^{1-\frac{1}{3}\lg x^2}\right)^2 = \left(\frac{1}{\sqrt[3]{100}}\right)^2;$ $\left(x^2\right)^{1-\frac{1}{3}\lg x^2} = \frac{1}{\sqrt[3]{10000}};$

Prologarithm with the base 10:

$\left(1-\frac{1}{3}\lg x^2\right)\lg x^2 = \lg(10)^{-\frac{4}{3}};$ $\left(1-\frac{1}{3}\lg x^2\right)\cdot\lg x^2 = -\frac{4}{3};$

We denote $\lg x^2 = y,$ then $\left(1-\frac{1}{3}y\right)y + \frac{4}{3} = 0;$ $y - \frac{1}{3}y^2 + \frac{4}{3} = 0 \Big| \cdot 3,$ $3y - y^2 + 4 = 0,$

$y^2 - 3y - 4 = 0.$ By Vieta's theorem, we have:

$y_1 = -1;$ $y_2 = 4.$ Returning to the replacement, we get:

$\lg x^2 = -1;$ $x^2 = 10^{-1};$

$x^2 = \frac{1}{10};$ $x_1 = -\frac{1}{\sqrt{10}};$ $x_1 = -\frac{1}{\sqrt{10}};$ $x_2 = \frac{1}{\sqrt{10}};$ $\lg x^2 = 4;$ $x^2 = 100;$ $x_3 = 100;$ $x_4 = -100.$

Answer: $\frac{1}{\sqrt{10}};$ $100.$

$\sqrt{\log_x \sqrt{5x}} = -\log_x 5.$

Solution:

$\sqrt{\log_x\left(\sqrt{5x}\right)^{\frac{1}{2}}} = -\log_x 5;$ $\sqrt{\frac{1}{2}\log_x 5 + \frac{1}{2}\log_x x} = -\log_x 5;$ $\sqrt{\frac{1}{2}\log_x 5 + \frac{1}{2}} = -\log_x 5;$

Let's square both sides:

$\frac{1}{2}\log_x 5 + \frac{1}{2} = \left(-\log_x 5\right)^2 \Big|: 2;$ $\log_5 x + 1 = 2\log^2{}_x 5;$

Substitution: $\log_x 5 = Z,$ then $2z^2 - z - 1 = 0;$ $\begin{cases} 2\log^2{}_x 5 - \log_x 5 - 1 = 0; \\ -\log_x 5 \geq 0. \end{cases}$

$D = 1 + 8 = 9.$ $Z = -\frac{1}{2};$ $Z = 1.$ $\begin{cases} \left[\begin{array}{l} \log_x 5 = -\frac{1}{2}, \\ \log_x 5 = 1, \end{array}\right. \\ \log_x 5 \leq 0. \end{cases}$

$x^{-1/2} = 5,$ $\left(x^{-\frac{1}{2}}\right)^{-2} = 5^{-2};$ $x = \frac{1}{25};$ $x = 5$ – It does not satisfy the inequality

$\log x^5 \leq 0.$

Answer: $\frac{1}{25}.$

$\log_x 25 = \frac{1}{\log_{25} x},$ then $2\cdot\frac{1}{\log_{25} x} - 3\log_{25} x = 1,$ $\frac{2}{\log_{25} x} - 3\log_{25} x = 1.$ Let's introduce a new variable: $\log_{25} x = y,$ $\frac{2}{y} - 3y = 1,$ $-3y^2 - y + 2 = 0,$ $3y^2 + y - 2 = 0,$

$D = 1 + 24 = 25 = 5^2.$ $y_1 = \dfrac{-1-5}{6} = -1;$ $y_2 = \dfrac{-1+5}{6} = \dfrac{2}{3}.$ $\log_{25} x = -1,$ $x = 25^{-1} = \dfrac{1}{25};$

$\log_{25} x = \dfrac{2}{3};$ $x = 25^{\frac{2}{3}} = \sqrt[3]{25^2} = \sqrt[3]{\left(5^2\right)^2} = \sqrt[3]{5^4} = \sqrt[3]{5^3 \cdot 5} = 5 \cdot \sqrt[3]{5}.$

Answer: $\dfrac{1}{25};$ $5 \cdot \sqrt[3]{5}.$

The mentioned formula is also useful in solving equations of the type:

$\log_x 3 + \log_{x^2} 9 + \log_{\sqrt{x}} \sqrt{3} = 1.$

Solution:

R.O.V.V.: $(0;\ 1) \cup (1;\ +\infty)$. We transform each term of the equation: $\log_x 3 = \dfrac{1}{\log_3 x};$

$\log_{x^2} 9 = 2\log_{x^2} 3 = \dfrac{2}{\log_3 x^2} = \dfrac{2}{2\log_3 x} = \dfrac{1}{\log_3 x};$ $\log_{\sqrt{x}} \sqrt{3} = \dfrac{1}{\log_{\sqrt{3}} \sqrt{x}} = \dfrac{1}{\log_{3^{\frac{1}{2}}} x^{\frac{1}{2}}} = \dfrac{1}{\log_3 x}.$

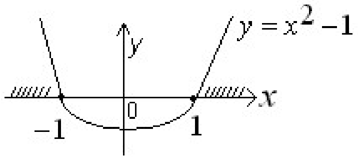
This equation will have the form: $\dfrac{1}{\log_3 x} + \dfrac{1}{\log_3 x} + \dfrac{1}{\log_3 x} = 1;$ $\dfrac{3}{\log_3 x} = 1,$ $\lg_3 x = 3,$

$x = 3^3 = 27.$ Answer: 27.

Or such an equation: $\log_3\left(2^{\sqrt{x^2-1}} + 1\right) = 2.$

Solution:

О.Д.З: $x^2 - 1 > 0,$ $(x-1)\cdot(x+1) > 0,$
$x \in (-\infty;\ -1) \cup (1;\ +\infty)$



By the definition of the logarithm of a number, we have:

$2^{\sqrt{x^2-1}} + 1 = 3^2;$ $2^{\sqrt{x^2-1}} = 9 - 1;$ $2^{\sqrt{x^2-1}} = 8;$ $2^{\sqrt{x^2-1}} = 2^3;$ $\sqrt{x^2 - 1} = 3;$ $x^2 - 1 = 9;$

$x^2 = 10;$ $x_1 = -\sqrt{10};$ $x_2 = \sqrt{10}.$

Answer: $-\sqrt{10};$ $\sqrt{10}.$

Self-study assignments:

The aforementioned identity is also necessary for solving logarithmic equations of the type:

8.1 $10^{\lg^2 x} + x^{\lg x} = 20.$ Answer: $0{,}1;$ $10.$

8.2 $\sqrt{\log_x 5\sqrt{5} + \log_{\sqrt{5}} 5\sqrt{5}} \cdot \log_{\sqrt{5}} x = -\sqrt{6}.$ Answer: $\dfrac{1}{5}.$

8.3 $\log_x x = \log_2\left(6 - x^2\right).$ Answer: $-3;$ $2.$

8.4 $x^{\log_x\left(x^2+3\right)} = -4.$ Answer: $x \in \varnothing.$

8.5 $\log_6(x-1) + \log_6(5x+3) = 2.$ Answer: $3.$

8.6 $\log_{\frac{1}{9}}\left(2x^2 - 2x - 1\right) = -\dfrac{1}{2}.$ Answer: $-1;$ $2.$

8.7 $2\log_3(x-2)-\log_3\left(x^2-4x+\dfrac{28}{9}\right)=2.$ Answer: 1; 3.

8.8 $\log_{x-1}9=2.$ Answer: 4.

8.9 $\lg\left(x^2-x-6\right)+x=\lg(x+2)+4.$ Answer: 4.

8.10 $x^{\log_2 x+2}=8.$ Answer: $\dfrac{1}{8}$.

8.11 $\log_x 9x^2\cdot\log^2{}_3 x=4.$ Answer: 3; $\dfrac{1}{9}$.

8.12 $\lg^2 x=1.$ Answer: 0,1; 10.

8.13 $\dfrac{1}{\log x-6}+\dfrac{5}{\log x+2}=1.$ Answer: 10^2; 10^8.

8.14 $\lg\left(x^{\lg x}\right)=1.$ Answer: 0,1; 10.

8.15 $\sqrt{2\lg(-x)}=\lg\sqrt{x^2}.$ Answer: -100; -1.

8.16 $\log_x(125x)\cdot\log^2{}_{25}x=1.$ Answer: 5; 5^{-4}.

8.17 $\lg(x+1)-\dfrac{1}{2}\lg(2x-3)-\lg\sqrt{x-3}=\dfrac{1}{2}-\lg\sqrt{2}.$ Answer: $\dfrac{11}{9}$; 4.

8.18 $x^{3-\lg\frac{200}{x}}=400.$ Answer: 0,01; 20.

8.19 $x^{3+\lg\frac{20}{x}}=8000.$ Answer: 0,001; 20.

8.20 $\sqrt{2\log_8(-x)}-\log_8\sqrt{x^2}=0.$ Answer: -1; -64.

8.21 $3^{\log^2{}_3 x}+x^{\log_3 x}=162.$ Answer: $\dfrac{1}{3}$; 9.

8.22 $2{,}5^{\log_3 x}+0{,}4^{\log_3 x}=2{,}9.$ Answer: $\dfrac{1}{3}$; 3.

8.23 $\log_x 9x^2\cdot\log^2{}_3 x=4.$ Answer: $\dfrac{1}{9}$; 3.

8.24 $\lg(\lg x)+\lg\left(\lg x^4-3\right)=0.$ Answer: $\dfrac{1}{\sqrt[4]{10}}$; 10.

8.25 $x^{\lg x-2}=1000.$ Answer: $\dfrac{1}{\sqrt[3]{100}}$; $\dfrac{1}{\sqrt[5]{100}}$.

8.26 $\log^2{}_5 x+3\log_5 x-4=0.$ Answer: 5; $\dfrac{1}{625}$.

8.27 $1+\log_2(x+5)=\log_2(3x-1)+\log_2(x-1).$ Answer: 3.

8.28 $x^{1+\lg x}=100.$ Answer: 0,01; 10.

8.29 $14^{\log_7 2}\cdot x^{\log_7 4x}=2.$ Answer: 0,5.